APPENDIX FOR *A SUPPLY AND DEMAND FRAMEWORK FOR TWO SIDED MATCHING MARKETS*

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Main Appendix

The appendix contains omitted proofs, as well as additional results which are used to derive the results in the text. Appendix A extends some results of classic matching theory to the continuum model. It proves existence of stable matchings, the lattice theorem, and the rural hospitals theorem. It also contains a proof of the supply and demand lemma. Appendix B derives the uniqueness and convergence of discrete economy results from Section 3. Appendix C collects additional omitted proofs.

Appendix A. Preliminary Results

We begin the analysis by deriving some basic properties of the set of stable matchings in the continuum model. Besides being of independent interest, they will be useful in the derivation of the main results. Throughout this section, unless otherwise stated, we fix a continuum economy $E = [\eta, S]$, and omit dependence on E, η, and S in the notation.

A.1. Monotonicity and Gross Substitutes Properties of Demand.

We note that demand functions satisfy some basic price-theoretic properties.

Definition A1. Aggregate demand for a set of colleges $C' \subseteq C$ is defined as

$$\sum_{c \in C'} D_c(P).$$

Aggregate demand for C' is **monotone** if this expression is monotone nonincreasing in $P_{c'}$ for all $c' \in C'$. Aggregate demand for C' satisfies the **gross substitutes** property if it is monotone nondecreasing in $P_{c''}$ for any $c'' \in C \backslash C'$. The following remark follows from the definition of demand.

Remark A1. Aggregate demand for any subset of colleges (and in particular for a single college) is monotone, satisfies the gross substitutes property, and is continuous in cutoffs.

A.2. Existence of Stable Matchings, Tâtonnement, and the Lattice Theorem.

We now establish the existence and lattice structure of stable matchings by using a tâtonnement procedure. The tâtonnement includes as particular cases procedures closely related to the Gale and Shapley student- and college- proposing algorithms. The map also includes as a particular case the map of Abdulkadiroğlu et al. (Forthcoming) in their setting, where η is uniform over scores and there is a measure 1 of students.

Given P_{-c} define the interval

$$I_c(P_{-c}) = \{p \in [0, 1] : D_c(p, P_{-c}) \leq S_c \text{ and } D_c(p, P_{-c}) = S_c \text{ if } p > 0\}.$$

That is, $I_c(P_{-c})$ is the set of cutoffs for college c that clear the market for c given the cutoffs of other colleges. Define the map $T(P)$ as

$$T_c(P) = \arg\min_{p \in I_c(P_{-c})} |p - P_c|.$$

That is, the map T has college c update its cutoff to the closest cutoff in $I_c(P_{-c})$ to P_c. This means that college c adjusts its cutoff as little as possible to clear the market for c, taking the cutoffs of other colleges as given. This map satisfies the following properties.

Proposition A1. *The map T is monotone non-decreasing (in the standard partial order of $[0,1]^C$), and the set of fixed points of T coincides with the set of market clearing cutoffs.*

Proof. We first show that T is well defined. Note that, because $D_c(1, P_{-c}) = 0$ and D is continuous, then either there exists $p \in [0,1]$ such that $D_c(p, P_{-c}) = S_c$ or $0 \in I_c(P_{-c})$. In either case, we have that $I_c(P_{-c})$ is nonempty. Note that, by monotonicity and continuity of demand, $I_c(P_{-c})$ is a compact interval.

We now show that T is monotone. To see this, consider $P \le P'$, $t_c = T_c(P)$, and $t'_c = T_c(P')$. To reach a contradiction, assume that $t'_c < t_c$. In particular $t_c > 0$. Using the monotonicity and gross substitutes properties we have that

$$S_c = D_c(t_c, P_{-c}) \le D_c(t'_c, P_{-c}) \le D_c(t'_c, P'_{-c}) \le S_c.$$

Likewise,

$$S_c = D_c(t_c, P_{-c}) \le D_c(t_c, P'_{-c}) \le D_c(t'_c, P'_{-c}) \le S_c.$$

From these inequalities we have that $D_c(t_c, P'_{-c}) = D_c(t'_c, P_{-c}) = S_c$. Hence,

$$[t'_c, t_c] \subseteq I_c(P_{-c}) \cap I_c(P'_{-c}).$$

The fact that the closest point to P_c in $I_c(P_{-c})$ is t_c implies that $P_c \ge t_c$. Therefore, $P'_c \ge t_c$. Therefore, $|t_c - P'_c| < |t'_c - P'_c|$, which contradicts $t'_c = T_c(P')$. This contradiction establishes that T is monotone.

Finally, we now show that the set of fixed points of T coincide with the market clearing cutoffs. This follows as P^* is a fixed point of T if and only if $P^*_c \in I_c(P^*_{-c})$ for all c, which is true if and only if P^* is a market clearing cutoff. $\square$

From Proposition A1 and Tarski's Theorem we have that stable matchings exist, as they correspond to fixed points of T.

Corollary A1. *At least one stable matching exists.*

Moreover, Tarski's Theorem and Proposition A1 also imply that the set of market clearing cutoffs is a lattice. Consider the sup ($\vee$) and inf ($\wedge$) operators in $[0,1]^C$ as lattice operators on cutoffs. That is, given an arbitrary set of cutoffs $X \subseteq 2^{([0,1]^C)}$, define

$$(\vee X)_c = \sup_{P \in X} P_c,$$

and analogously for the inf operator.

Theorem A1. *(Lattice Theorem) The set of market clearing cutoffs is a complete lattice.*

This theorem imposes a strict structure in the set of stable matchings. It differs from the classic Conway lattice theorem in the discrete setting (Knuth 1976), as the set of stable matchings forms a lattice with respect to the operation of taking the sup of the associated cutoff vectors. In the discrete model, where the sup of two matchings is defined as the matching where each student gets her favorite college in each of the matchings.

Finally, although this will not be used in the analysis that follows, we note that T produces algorithms closely related to deferred acceptance. Let $\mathbf{0}$ ($\mathbf{1}$) denote a C-dimensional vector of zeros (ones). Note that iterating the map T starting from cutoffs of $\mathbf{0}$ is similar (but not equal) to the student-proposing deferred acceptance algorithm. Namely, consider the sequence $(P^k)_{k \in \mathbb{N}}$ of cutoffs with $P^k = T^k(\mathbf{0})$. In the initial step $P^0 = \mathbf{0}$, the demand function $D^\theta(\mathbf{0})$ has each student point to her favorite college. In the second step, colleges raise their cutoffs to $P^1 = T(\mathbf{0})$. That is, all colleges reject just enough students to stay within capacity. The allocation given by $D^\theta(P^1)$ then allows rejected students to apply to more colleges, as long as e^θ is high enough.[1] This will make demand for each college weakly higher than supply, and the algorithm continues with more students being rejected and scores going up. An analogous argument shows that iterating T starting from $\mathbf{1}$ is closely related to the college-proposing deferred acceptance algorithm. The following Proposition shows that, much like deferred acceptance, these algorithms converge to the student-optimal and college-optimal stable matchings.

Proposition A2. *(Student- and college- optimal cutoff adjustment algorithms) The limits*

$$\begin{aligned} P^- &= \lim_{k \to \infty} T^k(\mathbf{0}) \\ P^+ &= \lim_{k \to \infty} T^k(\mathbf{1}). \end{aligned}$$

exist, and equal the smallest and largest market clearing cutoffs.

[1] Note that this is slightly different from the Gale and Shapley mechanism, where students can always apply to the next college in their preference list, as long as they have not been rejected, regardless of preferences. In contrast, in the present algorithm students are not allowed to apply to colleges where their scores are below current cutoffs. As such, the order of applications differs in the two procedures.

Proof. Consider first P^-. We have that $0 \leq T(0)$. Iterating this inequality, we have that $T^k(0) \leq T^{k+1}(0)$, so that the sequence $T^k(0)$ is monotone. Therefore, the limit P^- exists. The fact that P^- is a market clearing cutoff follows from continuity of demand. Namely, for every k, and $P^k = T^k(0)$,

$$D(P_c^{k+1}, P_c^k) \leq S_c,$$

with equality if $P_c^k > 0$. Taking the limit we have that

$$D_c(P^-) \leq S_c$$

with equality if $P_c^- > 0$. Finally, if P^* is a market clearing cutoff, then $0 \leq P^*$. Therefore,

$$T^k(0) \leq T^k(P^*) = P^*.$$

Taking the limit as $k \to \infty$, we get $P^- \leq P^*$. The proof for P^+ is analogous. □

A.3. **Rural Hospitals Theorem.** We now show that the rural hospitals theorem of classic matching theory extends to the continuum setting. This result implies that a college that does not fill its quota in one stable matching does not fill its quota in any other stable matching. Moreover, the measure of unmatched students is the same in every stable matching.

Theorem A2. ***(Rural Hospitals Theorem)*** *The measure of students matched to each college is the same in any stable matching. Furthermore, if a college does not fill its capacity, it is matched to the same set of students in every stable matching, except for a set of students with η measure 0.*

Proof. **Part 1: the measure of students matched to each college is the same in any stable matching.** Consider two market clearing cutoffs P and P', and let $P^+ = P \vee P'$. Take a college c, and assume without loss of generality that $P_c \leq P'_c$. By the gross substitutes property, we must have that $D_c(P^+) \geq D_c(P')$, as $P_c^+ = P'_c$ and the cutoffs of other colleges are higher under P^+. In addition, if $P'_c > 0$, then $D_c(P^+) = S_c \geq D_c(P)$. Moreover, if $P'_c = 0$, then $P_c = P'_c$, and $D_c(P^+) \geq D_c(P)$. Either way, we have that

$$D_c(P^+) \geq \max\{D_c(P), D_c(P')\}.$$

Moreover, the demand for staying unmatched $1 - \sum_{c \in C} D_c(\cdot)$ must at least as large under P^+ than under P or P'. Because demand for staying unmatched plus for all colleges always sums to 1, we have that, for all colleges, $D_c(P^+) = D_c(P) = D_c(P')$.

Part 2: a college that does not fill its capacity is matched to the same set of students in any stable matching.

Consider two stable matchings μ and μ'. Let $P = \mathcal{P}\mu$, $P' = \mathcal{P}\mu'$. Let $P^+ = P \vee P'$ and $\mu^+ = \mathcal{M}(P^+)$. Consider now a college c such that $\eta(\mu(c)) < S_c$. Therefore

$0 = P_c = P'_c = \max\{P_c, P'_c\} = P^+_c = 0$. By the definition of demand we have that $\mu(c) \subseteq \mu^+(c)$ and $\mu'(c) \subseteq \mu^+(c)$. By the first part of the theorem we know that the measure of $\mu(c), \mu'(c)$, and $\mu^+(c)$ are the same. Therefore, $\eta(\mu^+(c)\backslash\mu(c)) = 0$. Consequently, $\eta(\mu(c)\backslash\mu'(c)) \leq \eta(\mu^+(c)\backslash\mu'(c)) = 0$. Using a symmetric argument we get that $\eta(\mu'(c)\backslash\mu(c)) = 0$, completing the proof. □

Appendix B. Main Results

B.1. **Uniqueness.** We can now prove Theorem 1. The proof of Theorem 1 part 1 uses similar ideas as the proof of Lemma 4 in Abdulkadiroğlu et al. (Forthcoming). The first step in the proof is to note that the set of market clearing cutoffs is a lattice. Abdulkadiroğlu et al. (Forthcoming) prove this with a particular case of the map used in our proof of the lattice theorem. The second step is to show that demand for a certain subset of colleges is the same under the largest and smallest market clearing cutoffs. This is true by definition in Abdulkadiroğlu et al.'s model, while in our model it depends on the rural hospitals theorem. The third part shows that, with full support, this implies that the smallest and largest market clearing cutoffs are equal. This third step is essentially the same as in Abdulkadiroğlu et al.'s proof.

Denote the excess demand given a vector of cutoffs P and an economy $E = [\eta, S]$ by

$$z(P|E) = D(P|\eta) - S.$$

Proof. (Theorem 1)

Part (1):

By the lattice theorem, E has smallest and greatest market clearing cutoffs $P^- \leq P^+$, and corresponding stable matchings μ^-, μ^+. Let $C^+ = \{c \in C : P^+_c \neq P^-_c\}$. In particular, for all colleges in C^+ we have $P^+_c > 0$. Let $C^0 = C\backslash C^+$. Note that, since for all colleges $c \in C^0$ we have $P^+_c = P^-_c$, and for all colleges c in C^+ we have $P^+_c > P^-_c$, we have that

$$\{\theta \in \Theta : \mu^+(\theta) \in C^+\} \subseteq \{\theta \in \Theta : \mu^-(\theta) \in C^+\}.$$

By the rural hospitals theorem, the difference between these two sets has measure 0. That is

$$\eta(\{\theta \in \Theta : \mu^-(\theta) \in C^+\}\backslash\{\theta \in \Theta : \mu^+(\theta) \in C^+\}) = 0.$$

Let $\succ^+$ be a fixed preference relation that ranks all colleges in C^+ higher than those in C^0. Then the set in the above equation must contain all students with preference $\succ^+$ and scores $P^-_c \leq e^\theta_c < P^+_c$ for all $c \in C^+$. That is,

$$\begin{aligned}&\{(\succ^+, e^\theta) \in \Theta : P^-_c \leq e^\theta_c < P^+_c \text{ for all } c \in C^+\}\\ \subseteq\ &\{\theta \in \Theta : \mu^-(\theta) \in C^+\}\backslash\{\theta \in \Theta : \mu^+(\theta) \in C^+\}.\end{aligned}$$

Therefore, the former set has measure 0:

$$\eta(\{(\succ^+, e^\theta) \in \Theta : P_c^- \le e_c^\theta < P_c^+ \text{ for all } c \in C^+\}) = 0.$$

By the full support assumption, and since $P_c^- < P_c^+$ for all c in C^+, this can only be the case if C^+ is the empty set.[2] This implies that $P^- = P^+$, and therefore there exists a unique vector of market clearing cutoffs.

Part (2):

The proof follows from two claims. The first claims shows that, in an economy with more than one stable matching, either demand is not differentiable, or the derivative matrix of demand is not invertible. The proof of the claim follows from the observation that demand is constant for cutoffs between the smallest and largest market clearing cutoffs. This observation is a consequence of monotonicity of demand and of the rural hospitals theorem.

Claim B1. Consider S such that $E = [\eta, S]$ has more than one stable matching. Then there exists at least one market clearing cutoff P^* of E where either demand is not differentiable, or the derivative matrix $\partial D(P^*|E)$ of the demand function is singular.

Proof of Claim B1. If there is at least one market clearing cutoff of E where demand is not differentiable, we are done. Consider now the case where demand is differentiable at all market clearing cutoffs.

By the lattice theorem, economy E has smallest and largest market clearing cutoffs, with $P^- \le P^+$. Let

$$C^+ = \{c \,:\, P_c^- < P_c^+\}.$$

C^+ is nonempty, due to the assumption that E has more than one market clearing cutoff. Let F be the subspace of $\mathbb{R}^C$ where all coordinates corresponding to colleges not in C^+ are zero, that is

$$F = \{v \in \mathbb{R}^C : v_c = 0 \text{ for all } c \notin C^+\}.$$

Consider $P \in [P^-, P^+]$. For any college $c \notin C^+$ we have $P_c^+ = P_c^- = P_c$, and therefore, by the gross substitutes property,

$$D_c(P^+|\eta) \ge D_c(P|\eta) \ge D_c(P^-|\eta).$$

By the rural hospitals theorem, $D_c(P^-|\eta) = D_c(P^+|\eta)$, and therefore $D_c(\cdot|\eta)$ is constant in the cube $[P^-, P^+]$. In particular, for any $c \notin C^+$ and $c' \in C^+$ we have that

$$\partial_{c'} D_c(P^-|\eta) = 0. \tag{1}$$

[2]Note that this argument works more generally than with the full support assumption. For example, if college preferences were perfectly correlated and the support of student types was equal to $\{\theta : e_c^\theta = e_{c'}^\theta \text{ for all } c, c' \in C\}$, then the fact that the set above has measure 0 would also imply that there exists a single vector of market clearing cutoffs.

That is, the derivative matrix ∂D takes the subspace F into itself.

In addition, for all $P \in [P^-, P^+]$, it follows from the monotonicity property of aggregate demand that

$$\sum_{c \in C^+} D_c(P^-|\eta) \geq \sum_{c \in C^+} D_c(P|\eta) \geq \sum_{c \in C^+} D_c(P^+|\eta).$$

By the rural hospitals theorem, we have that $D_c(P^-|\eta) = D_c(P^+|\eta)$ for all $c \in C^+$, and therefore $\sum_{c \in C^+} D_c(P|\eta)$ is constant on the cube $[P^-, P^+]$. This implies that

$$\sum_{c \in C^+} \partial_{c'} D_c(P^-|\eta) = 0$$

for all $c, c' \in C^+$. Consequently, the linear transformation $\partial D(P^-|\eta)$ restricted to the subspace F is not invertible. Because $\partial D(P^-|\eta)$ takes F into itself, we have that $\partial D(P^-|\eta)$ is not invertible, proving the claim.

□

We now note that, as a consequence of Sard's Theorem,[3] for almost all vectors S, the demand function is continuously differentiable with a nonsingular derivative at all market clearing cutoffs. The intuition for this claim is similar to the observation that, in a standard supply and demand model with a single good, demand and supply almost never cross at a point where demand is vertical.

Claim B2. For almost every $S \in \mathbb{R}^C_+$ with $\sum_c S_c < 1$, at every market clearing cutoff P^* of $[\eta, S]$, demand is continuously differentiable and the derivative matrix $\partial D(P^*|\eta)$ is invertible.

Proof of Claim B2. The assumption that $\sum_c S_c < 1$ implies that all market clearing cutoffs P^* of $[\eta, S]$ satisfy $D(P^*|\eta) = S$. Moreover, all market clearing cutoffs are strictly greater than 0. The assumption that supply of all colleges is strictly positive implies that market clearing cutoffs are strictly lower than 1. Therefore, any vector of market clearing cutoffs is in the open set $(0, 1)^C$.

Define the closure of the set of points where the demand function is not differentiable as

$$NDP = \text{closure}(\{P \in (0, 1)^C : D(\cdot|\eta) \text{ is not continuosuly differentiable at } P\}).$$

Note that, by the definition of a regular distribution of types η, the image of NDP under $D(\cdot|\eta)$ has measure 0. In particular, for almost every S, demand at every associated market clearing cutoff is continuously differentiable.

[3]Sard's theorem can be stated as follows. Let $f : X \to \mathbb{R}^C$ be a continuously differentiable function from an open set $X \subseteq \mathbb{R}^C$ into $\mathbb{R}^C$. A critical point of f is $x \in X$ such that $\partial f(x)$ is a singular matrix. A critical value is any $y \in \mathbb{R}^C$ that is the image of a critical point. Sard's theorem states that the set of critical values of f has Lebesgue measure 0. See Guillemin and Pollack (1974); Milnor (1997).

Moreover, restricted to the open set $(0,1)^C \backslash NDP$, the demand function is continuously differentiable. Consequently, by Sard's theorem, the set of critical values of $D(\cdot|\eta)$ restricted to $(0,1)^C \backslash NDP$ has measure 0. That is, for almost all S, there are no vectors P in $(0,1)^C \backslash NDP$ such that $D(P|\eta) = S$ and $\partial D(P|\eta)$ is singular. Taken together, these two observations imply that, for almost all S, demand at associated market clearing cutoffs is both continuously differentiable and has an invertible derivative matrix.

□

The result now follows from Claims B2 and B1. Take S such that $\sum_c S_c < 1$ and economy $E = [\eta, S]$ has more than one stable matching. By Claim B1, in at least one of the market clearing cutoffs of E, demand is either non-differentiable, or has a singular derivative. However, Claim B2 shows that this only holds for a measure 0 set of vectors S. Therefore, the set of vectors S such that $\sum_c S_c < 1$ and there is more than one stable matching has measure 0, completing the proof. □

B.2. **Continuity and convergence.**

B.2.1. *Continuity Within $\mathcal{E}$.* This section establishes that the stable matching correspondence is continuous around an economy $E \in \mathcal{E}$ with a unique stable matching. That is, that if a continuum economy has a unique stable matching, it varies continuously with the fundamentals.

Note that, by our definition of convergence, we have that if the sequence of continuum economies $\{E^k\}_{k\in\mathbb{N}}$ converges to a continuum economy E, then the functions $z(\cdot|E^k)$ converge pointwise to $z(\cdot|E)$. Moreover, using the assumption that firms' indifference curves have measure 0 at E, we have that, if we simultaneously take convergent sequences of cutoffs and economies, then the associated excess demand converges to excess demand in the limit. This is formalized in the following lemma.

Lemma B1. *Consider a continuum economy $E = [\eta, S]$, a vector of cutoffs P and a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P. If $\{\eta^k\}_{k\in\mathbb{N}}$ converges to η in the weak-* sense and $\{S^k\}_{k\in\mathbb{N}}$ converges to S then*

$$z(P^k|[\eta^k, S^k]) = D(P^k|\eta^k) - S^k$$

converges to $z(P|E)$.

Proof. Let G^k be the set

$$G^k = \cup_c\{\theta \in \Theta : \|e_c^\theta - P_c\| \leq \sup_{k'\geq k} \|P_c^{k'} - P_c\|\}.$$

The set

$$\cap_k G^k = \cup_c\{\theta \in \Theta : e_c^\theta = P_c\},$$

has η-measure 0 by the strict preferences assumption 1. Since the G^k are nested, we have that $\eta(G^k)$ converges to 0 as $k \to \infty$.

Now take $\epsilon > 0$. There exists k_0 such that for all $k \geq k_0$ we have $\eta(G^k) < \epsilon/4$. Since the measures η^k converge to η in the weak sense, we may assume also that $\eta^k(G^{k_0}) < \epsilon/2$. Since the G^k are nested, this implies $\eta^k(G^k) < \epsilon/2$ for all $k \geq k_0$. Note that $D^\theta(P)$ and $D^\theta(P^k)$ may only differ for $\theta \in G^k$. We have that

$$\|D(P|\eta) - D(P^k|\eta^k)\| \leq \|D(P|\eta) - D(P|\eta^k)\| + \|D(P|\eta^k) - D(P^k|\eta^k)\|.$$

As η^k converges to η, we may take k_0 large enough so that the first term is less than $\epsilon/2$. Moreover, since the measure $\eta(G^k) < \epsilon/2$, we have that for all $k > k_0$ the second term is less than $\epsilon/2$. Therefore, the above difference is less than ϵ, completing the proof. □

Note that this lemma immediately implies the following:

Lemma B2. *Consider a continuum economy $E = [\eta, S]$, a vector of cutoffs P a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P, and a sequence of continuum economies $\{E^k\}_{k\in\mathbb{N}}$ converging to E. We have that $z(P^k|E^k)$ converges to $z(P|E)$.*

Using the lemma, we show that the stable matching correspondence is upper hemicontinuous.

Proposition B1. ***(Upper Hemicontinuity)*** *The stable matching correspondence is upper hemicontinuous*

Proof. Consider a sequence $\{E^k, P^k\}_{k\in\mathbb{N}}$ of continuum economies and associated market clearing cutoffs, with $E^k \to E$ and $P^k \to P$, for some continuum economy E and vector of cutoffs P. We have $z(P|E) = \lim_{k\to\infty} z(P^k, E^k) \leq 0$. If $P_c > 0$, for large enough k we must have $P_c^k > 0$ so that $z_c(P|E) = \lim_{k\to\infty} z_c(P^k, E^k) = 0$. Therefore, P is a market clearing cutoff of E. □

With uniqueness, continuity also follows easily.

Lemma B3. ***(Continuity)*** *Let E be a continuum economy with a unique stable matching. Then the stable matching correspondence is continuous at E.*

Proof. Let P be the unique market clearing cutoff of E. Consider a sequence $\{E^k, P^k\}_{k\in\mathbb{N}}$ of economies and associated market clearing cutoffs, with $E^k \to E$. Assume, by contradiction that P^k does not converge to P. Then P^k has a convergent subsequence that converges to another point $P' \in [0,1]^C$, with $P' \neq P$. By the previous proposition, P' must be a market clearing cutoff of E, contradicting the fact that P is the unique market clearing cutoff of E. □

B.2.2. *Convergence of Finite Economics.* We now consider the relationships between the stable matchings of a continuum economy, and stable matchings of a sequence of discrete economies that converge to it. The argument follows similar lines as that for convergence of a sequence of continuum economies in the preceding Subsection.

For finite economies F, we define the excess demand function as in the continuous case:

$$z(P|F) = D(P|F) - S.$$

Note that, with this definition, P is a market clearing cutoff for finite economy F iff $z(P|F) \leq 0$, with $z_c(P|F) = 0$ for all colleges c such that $P_c > 0$.

From Lemma B1 we immediately obtain the following result.

Lemma B4. *Consider a limit economy E, a sequence of cutoffs $\{P^k\}_{k\in\mathbb{N}}$ converging to P, and a sequence of finite economies $\{F^k\}_{k\in\mathbb{N}}$ converging to E. We then have that $z(P^k|F^k)$ converges to $z(P|E)$.*

This lemma then implies the following upper hemicontinuity property.

Proposition B2. ***(Convergence)*** *Let E be a continuum economy, and $\{F^k, P^k\}_{k\in\mathbb{N}}$ a sequence of discrete economies and associated market clearing cutoffs, with $F^k \to E$ and $P^k \to P$. Then P is a market clearing cutoff of E.*

Proof. **(Proposition B2)** We have $z(P|E) = \lim_{k\to\infty} z(P^k|F^k) \leq 0$. If $P_c > 0$, then $P_c^k > 0$ for large enough k, and we have $z_c(P|E) = \lim_{k\to\infty} z_c(P^k|F^k) = 0$. □

When the continuum economy has a unique stable matching, we can prove the stronger result below.

Lemma B5. ***(Convergence with uniqueness)*** *Let E be a continuum economy with a unique market clearing cutoff P, and $\{F^k, P^k\}_{k\in\mathbb{N}}$ a sequence of discrete economies and associated market clearing cutoffs, with $F^k \to E$. Then $P^k \to P$.*

Proof. **(Lemma B5)** To reach a contradiction, assume that P^k does not converge to P. Then P^k has a convergent subsequence that converges to another point $P' \in [0,1]^C$, with $P' \neq P$. By Proposition B2, P' is a market clearing cutoff of E. Therefore, we have that $P' \neq P$ is a market clearing cutoff, a contradiction with P being the unique market clearing cutoff of E. □

B.2.3. *Proof of Theorem 2.* Theorem 2 follows from the previous results.

Proof. **(Theorem 2)** Part (3) follows from Lemma B3 and Part (1) follows from Lemma B2. As for Part (2), note first that given an economy F^k the set of market clearing cutoffs is compact, which follows from the definition of market clearing cutoffs, and continuity of the demand function. Therefore, there exist market clearing cutoffs P^k

and P'^k of F^k such that the diameter of F^k is $\|P^k - P'^k\|$. However, by Part (1), both sequences $\{P^k\}_{k\in\mathbb{N}}$ and $\{P'^k\}_{k\in\mathbb{N}}$ are converging to P, and therefore the diameter of F^k is converging to 0. $\square$

Appendix C. Additional Proofs

This section collects proofs that were omitted in the text.

C.1. **The Supply and Demand Lemma.** We first prove the continuous version of the supply and demand lemma.

Proof. **(Lemma 1)** Let μ be a stable matching, and $P = \mathcal{P}\mu$. Consider a student θ with $\mu(\theta) = c$. By definition of the operator $\mathcal{P}$, $P_c \leq e^\theta_c$. Consider a college c' that θ prefers over c. By the open on the right condition, there is a student $\theta_+ = (\succ^\theta, e^{\theta+})$ with slightly higher scores than θ who is matched to c and prefers c'. By stability of μ, all the students that are matched to c' have higher c' scores than θ_+, so $P_{c'} \geq e^{\theta+}_{c'} > e^\theta_{c'}$. Following the argument for all colleges that θ prefers to c, we see that there are no colleges that are better than c and that θ can afford at cutoffs P. Therefore, c is better than any other college that θ can afford, so $D^\theta(P) = \mu(\theta)$. This implies that no college is over-demanded given P, and that $\mathcal{MP}\mu = \mu$. To conclude that P is a market clearing cutoff, note that if $\eta(\mu(c)) < S_c$ stability implies that a student whose first choice is c and has score at c of zero is matched to c. Therefore, $P_c = 0$.

To prove the other direction of the lemma, let P be a market clearing cutoff, and $\mu = \mathcal{M}P$. By the definition of $D^\theta(P)$, μ is open on the right and measurable. Because P is a market clearing cutoff, μ respects capacity constraints. It respects the consistency conditions to be a matching by definition. To show that μ is stable, consider any potential blocking pair (θ, c) with $\mu(\theta) \prec^\theta c$. Since θ does not demand c (i.e., $\mu(\theta) = D^\theta(P) \neq c$), it must be that $P_c > e^\theta_c$, so $P_c > 0$ and c has no empty seats. For any type θ' such that $\theta' \in \mu(c)$, we have that $e^{\theta'}_c \geq P_c > e^\theta_c$, and therefore (θ, c) is not a blocking pair. Thus, μ is stable.

We now show that $\mathcal{PM}$ is the identity. Let $P' = \mathcal{P}\mu$. If $\mu(\theta) = c$, then $e^\theta_c \geq P_c$. Therefore,

$$P'_c = (\mathcal{P}\mu)_c = \inf_{\theta\in\mu(c)} e^\theta_c \geq P_c.$$

However, if θ is a student with $e^\theta_c = P_c$ whose favorite college is c, then $\mu(\theta) = D^\theta(P) = c$. Therefore $P'_c \leq P_c$. These two inequalities imply that $P' = P$, and therefore $\mathcal{PM}P = P$. $\square$

We now prove the discrete version of the supply and demand Lemma.

Proof. **(Lemma 2)** Consider a stable matching $\tilde{\mu}$, and let $P = \tilde{\mathcal{P}}\tilde{\mu}$. Any student θ who is matched to a college $c = \tilde{\mu}(\theta)$ can afford her match, as $P_c \leq e^\theta_c$ by the definition of

$\tilde{\mathcal{P}}$. Likewise, students who are unmatched may always afford being unmatched. Note that no student can afford a college $c' \succ^\theta \tilde{\mu}(\theta)$: if she could, then $P_{c'} \leq e^\theta_{c'}$, and by the definition of $\tilde{\mathcal{P}}$, there would be another student θ' matched to c' with $e^{\theta'}_{c'} < e^\theta_{c'}$, or empty seats at c', which would contradict $\tilde{\mu}$ being stable. Consequently, no student can afford any option better than $\tilde{\mu}(\theta)$, and all students can afford their own match $\tilde{\mu}(\theta)$. This implies $D^\theta(P) = \tilde{\mu}(\theta)$. This proves both that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ is the identity, and that P is a market clearing cutoff.

In the other direction, let P be a market clearing cutoff, and $\tilde{\mu} = \tilde{\mathcal{M}}P$. By the definition of the operator $\tilde{\mathcal{M}}$ and the market clearing conditions, $\tilde{\mu}$ is a matching, so we only have to show there are no blocking pairs. Assume by contradiction that (θ, c) is a blocking pair. If c has empty slots, then $P_c = 0 \leq e^\theta_c$. If c is matched to a student θ' that is less preferred than θ, then $P_c \leq e^{\theta'}_c \leq e^\theta_c$. Hence, we must have $P_c \leq e^\theta_c$. However, this implies that $c \preceq^\theta D^\theta(P) = \tilde{\mu}(\theta)$, so (θ, c) cannot be a blocking pair, reaching a contradiction. □

C.2. **School Competition.** We now prove the proposition for the school competition application in Section 4.1.

Proof. (Proposition 2)

Aggregate quality is defined as

$$\begin{aligned} Q_c(\delta) &= \int_{\mu_\delta(c)} e^\theta_c \, d\eta_\delta(\theta) \\ &= \int_{\{\theta : D^\theta(P^*(\delta)) = c\}} e^\theta_c \cdot f_\delta(\theta)\, d\theta. \end{aligned}$$

By Leibniz's rule, Q_c is differentiable in δ_c, and the derivative is given by

$$\begin{aligned} \frac{dQ_c(\delta)}{d\delta_c} = \; & \textstyle\int_{\{\theta : D^\theta(P^*(\delta)) = c\}} e^\theta_c \cdot \frac{d}{d\delta_c} f_\delta(\theta)\, d\theta \\ + \; & \textstyle\sum_{c' \neq c} \frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \cdot \bar{P}_{c'c} \\ - \; & \textstyle\frac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c' \neq c} M_{cc'}] \cdot P^*_c. \end{aligned} \tag{2}$$

The first term is the integral of the derivative of the integrand, and the last two terms the change in the integral due to the integration region $\{\theta : D^\theta(P^*(\delta)) = c\}$ changing with δ_c. The terms in the second line are the changes due to changes in the cutoffs $P^*_{c'}$, the students that school c gains (or loses) because school c' becomes more (less) selective. The quantity of these students is $\frac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c}$, and their average quality $\bar{P}_{c'c}$. The last line is the term representing the students lost due to school c raising its cutoff P_c. These students number $[M_{c\emptyset} + \sum_{c' \neq c} M_{c'c}]$, and have average quality P^*_c. Note that, since the total number of students admitted at school c is constant and equal to S_c, we have

$$\begin{aligned} 0 = \quad & \int_{\{\theta: D^\theta(P^*(\delta))=c\}} \tfrac{d}{d\delta_c} f_\delta(\theta)\, d\theta \\ + \quad & \sum_{c'\neq c} \tfrac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \\ - \quad & \tfrac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c'\neq c} M_{cc'}]. \end{aligned}$$

Therefore, if we substitute $\frac{dP^*_c}{d\delta_c} \cdot [M_{c\emptyset} + \sum_{c'\neq c} M_{cc'}]$ in equation (2) we have

$$\begin{aligned} \frac{dQ_c(\delta)}{d\delta_c} = \quad & \int_{\{\theta: D^\theta(P^*(\delta))=c\}} [e^\theta_c - P^*_c] \cdot \tfrac{d}{d\delta_c} f_\delta(\theta)\, d\theta \\ + \quad & \sum_{c'\neq c} \tfrac{dP^*_{c'}}{d\delta_c} \cdot M_{c'c} \cdot [\bar{P}_{c'c} - P^*_c]. \end{aligned}$$

The term in the second line is the market power effect as defined in the text. That the term in the first line equals the expression in Proposition 2 follows from the definition of N_c and $\bar{e}_c$.

To see that the direct effect is positive, note that by definition $\bar{e}_c \geq P^*(\delta)$, and since $u^i_c(\delta)$ is increasing in δ_c we have $N_c \geq 0$. □

We now provide the derivation of the market power effect in Section 4.1 when schools are symmetrically differentiated.

Additional details on Section 4.1.

In Section 4.1 we gave a formula for the market power effect when the function $f_\delta(\theta)$ is symmetric over all schools, and schools choose the same level of quality. This formula follows from substituting an expression for $dP^*/d\delta_c$ in the formula for the market power term. To obtain the formula for $dP^*/d\delta_c$, we start from the point δ where all $\delta_c = \delta_{c'}$. In this case, all $P^*_c(\delta) = P^*_{c'}(\delta)$. If school c changes δ_c, the the cutoff $P^*_c(\delta)$ of school c will change. The cutoffs of the other schools will change, but all other schools $c' \neq c$ will have the same cutoff $P^*_{c'}(\delta)$. We denote $D_c(P_c, P_{c'}|\delta)$ for the demand for school c, and $D_{c'}(P_c, P_{c'}|\delta)$ for the demand for each other school under these cutoffs. Applying the implicit function theorem to the system of two equations

$$\begin{aligned} D_c(P_c, P_{c'}|\delta) &= S_c \\ D_{c'}(P_c, P_{c'}|\delta) &= S_c \end{aligned}$$

we get

$$\begin{aligned} \frac{d}{dP_c} D_c \cdot \frac{dP_c}{d\delta_c} + \frac{d}{dP_{c'}} D_c \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c} D_c &= 0 \\ \frac{d}{dP_c} D_{c'} \cdot \frac{dP_c}{d\delta_c} + \frac{d}{dP_{c'}} D_{c'} \cdot \frac{dP_{c'}}{d\delta_c} + \frac{d}{d\delta_c} D_{c'} &= 0. \end{aligned}$$

Substituting the derivative of the demand function as a function of the mass of agents on the margins $\tilde{M}_{cc'}$, the system becomes

$$\begin{aligned} -[M_{c\emptyset}+(C-1)M_{cc'}]\cdot\frac{dP_c}{d\delta_c}+[(C-1)M_{c'c}]\cdot\frac{dP_{c'}}{d\delta_c}+\frac{d}{d\delta_c}D_c &= 0 \\ [M_{cc'}]\cdot\frac{dP_c}{d\delta_c}-[M_{c'\emptyset}+M_{c'c}]\cdot\frac{dP_{c'}}{d\delta_c}+\frac{d}{d\delta_c}D_{c'} &= 0. \end{aligned}$$

Due to the symmetry of the problem, $M_{cc'}=M_{c'c}$, $M_{c\emptyset}=M_{c'\emptyset}$, and $\frac{d}{d\delta_c}D_c=-(C-1)\frac{d}{d\delta_c}D_{c'}$. The formula in the text then follows from solving the system.

C.3. **Convergence Rates for Random Economies.** We begin by bounding the difference between market clearing cutoffs in a continuum economy and in a finite approximation.

Proposition C1. *Assume that the continuum economy $E=[\eta,S]$ admits a unique stable matching μ, and $\sum_c S_c<1$. Let P^* be the associated market clearing cutoff, and assume $D(\cdot|\eta)$ is C^1, and $\partial D(P^*)$ is invertible. Then there exists $\alpha\geq 0$ such that, for any finite economy $F=[\eta^F,S^F]$, we have*

$$\begin{aligned} &\sup\{||P^F-P^*||: P^F \text{ is a market clearing cutoff of } F\} \\ &\quad\leq \alpha\cdot(\sup_{P\in[0,1]^C}||D(P|\eta)-D(P|\eta^F)||+||S-S^F||). \end{aligned}$$

The proposition shows that the distance between market clearing cutoffs of a continuum economy and a discrete approximation is of the same order of magnitude as the distance between the associated vectors of capacities, plus the difference between the demand functions. Therefore, the continuum model is a good approximation as long as the distance between the empirical distribution of types η^F and η and the per capita supply vectors S^F and S is small.

Proof. Note that since $\sum_c S_c<1$, market clearing cutoffs satisfy $z(P|E)=0$. In what follows we always take α to be large enough such that, for any finite economy F such that the bound in the proposition has any content (that is, the right side is less than one), $\sum_c S_c^F<1$. This guarantees that market clearing cutoffs in such an economy must satisfy $z(P|F)=0$.

The proof begins by showing that at economy E, cutoffs P that are far from the market clearing cutoff P^* have large excess demands, in the sense that their norm is bounded below by a multiple of the distance to the market clearing cutoff P^*. Formally, let $B^\epsilon=\{P\in[0,1]^C:\|P-P^*\|<\epsilon\}$. Let

$$\begin{aligned} P^\epsilon &= \arg\min_{P\notin B^\epsilon}||z(P|E)|| \text{ and} \\ M^\epsilon &= \min_{P\notin B^\epsilon}||z(P|E)||. \end{aligned}$$

Note that, due to the continuity of the demand function, both P^ϵ and M^ϵ are well defined. Moreover, P^ϵ may be a set of values, in the case of multiple minima. In what follows, we will take a single-valued selection from this set, so that P^ϵ represents one of the minima. With this convention, $M^\epsilon = \|z(P^\epsilon|E)\|$. We will now show that there exists $\alpha > 0$ such that for all $0 < \epsilon \leq 1$

$$M^\epsilon \geq \frac{1}{\alpha} \cdot \epsilon.$$

To see this, note that since $D(\cdot|E)$ is C^1, we have that

$$z(P|E) = D(P|E) - S = \partial D(P^*|E) \cdot (P - P^*) + g(P - P^*), \tag{3}$$

where the continuous function $g(\cdot)$ satisfies that for any $\epsilon' > 0$, there exists $\delta > 0$ such that, for all $P \in B^\delta$,

$$\frac{\|g(P - P^*)\|}{\|P - P^*\|} < \epsilon'.$$

Since $\partial_P D(P^*|E)$ is nonsingular, we may take $A > 0$ such that

$$||\partial_P D(P^*|E) \cdot v|| \geq 2A \cdot ||v||, \tag{4}$$

for any vector $v \in \mathbb{R}^C$.

By the property of $g(\cdot)$ above, with $\epsilon' = A$, we may take $0 < \epsilon_0 \leq 1$ such that

$$\frac{||g(P - P^*)||}{||P - P^*||} < A \tag{5}$$

for all $P \in B^{\epsilon_0}$. Therefore, for all $P \in B^{\epsilon_0}$ we have

$$\begin{aligned}
||z(P|E)|| &= ||\partial D(P^*|E) \cdot (P - P^*) + g(P - P^*)|| \\
&\geq ||\partial D(P^*|E) \cdot (P - P^*)|| - ||g(P - P^*)|| \\
&\geq 2A \cdot ||P - P^*|| - ||\frac{g(P - P^*)}{||P - P^*||}|| \cdot ||P - P^*|| \\
&\geq 2A \cdot ||P - P^*|| - A \cdot ||P - P^*|| \\
&= A \cdot ||P - P^*||.
\end{aligned}$$

The first equality follows from the derivative formula for excess demand in equation (3). The inequality in the second line follows from the triangle inequality. The inequality in the third line follows from the bound in inequality (4) for the left term, and algebra for the right term. The inequality in the fourth line is a consequence of applying the bound in inequality (5) to the right term. Finally, the last line follows from subtracting the right term from the left term. The above reasoning establishes that for all $P \in B^{\epsilon_0}$ excess demand is bounded from below by

$$\|z(P|E)\| \geq A \cdot \|P - P^*\|,$$

which is linear in $\|P - P^*\|$. In particular, this implies that, for all $0 < \epsilon < \epsilon_0$, we have

$$M^\epsilon \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\}. \tag{6}$$

We will now use this bound to obtain a bound that is valid for all $0 < \epsilon \leq 1$. Since E has a unique stable matching we have that $M^{\epsilon_0} > 0$. Take $\alpha > 0$ such that

$$\frac{1}{\alpha} = \min\{A, M^{\epsilon_0}\}.$$

Therefore, if $0 < \epsilon < M^{\epsilon_0}/A$ we have $M^\epsilon \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\} = A \cdot \epsilon \geq \frac{1}{\alpha} \cdot \epsilon$. If $M^{\epsilon_0}/A \leq \epsilon \leq 1$, then $M^\epsilon \geq \min\{A \cdot \epsilon, M^{\epsilon_0}\} = M^{\epsilon_0} \geq \frac{1}{\alpha} \geq \frac{1}{\alpha}\epsilon$. Either way, we have the desired bound

$$M^\epsilon \geq \frac{1}{\alpha} \cdot \epsilon \tag{7}$$

for all $0 < \epsilon \leq 1$.

We now prove the proposition. If P^F is a market clearing vector of the finite economy F then

$$\|z(P^F|E) - z(P^F|F)\| = \|z(P^F|E)\| \geq \frac{1}{\alpha} \cdot \|P^F - P^*\|.$$

The first equality follows from excess demand at a strictly positive market clearing cutoff being 0, and the second by the bound for M^ϵ in inequality (7). Moreover, by the triangle inequality we have that

$$\begin{aligned} ||z(P^F|E) - z(P^F|F)|| &\leq ||D(P^F|\eta) - D(P^F|\eta^F)|| + ||S - S^F|| \\ &\leq \sup_{P \in [0,1]^C} ||D(P|\eta) - D(P|\eta^F)|| + ||S - S^F||. \end{aligned}$$

Combining these two inequalities we obtain the desired bound

$$\|P^F - P^*\| \leq \alpha \cdot (\sup_{P \in [0,1]^C} \|D(P|\eta) - D(P|\eta^F)\| + \|S - S^F\|).$$

□

Using this bound, we can prove the results on sequences of randomly drawn finite economies.

Proof. (Proposition 3)

Part (1): Almost sure convergence.

First we show that that the sequence of random economies $\{F^k\}_{k \in \mathbb{N}}$ converges to E almost surely. It is true by assumption that S^k converges to S. Moreover, by the Glivenko-Cantelli Theorem, the realized measure η^k converges to η in the weak-* topology almost surely. Therefore, by definition of convergence, we have that F^k converges to E almost surely. This implies, by Theorem 2, that μ^k converges to μ.

Part (2): Bound on $\|P^* - P^k\|$.

By Proposition C1, we may take $\alpha_0 \geq 0$ such that, for all k, realization of the discrete economy F^k, and market clearing cutoff P^k of F^k,

$$\|P^k - P^*\| \leq \alpha_0 \cdot (\sup_{P\in[0,1]^C} \|D(P|\eta) - D(P|\eta^k)\| + \|S - S^k\|). \tag{8}$$

Let the agents in finite economy F^k be $\theta^{1,k}, \theta^{2,k}, \dots, \theta^{k,k}$. The demand function at economy F^k is the random variable

$$D_c(P|\eta^k) = \sum_{i=1,\dots,k} 1_{\theta^{i,k}\in\{\theta\in\Theta: D^\theta(P)=c\}}/k.$$

That is, D_c are similar to empirical distribution functions, measuring the fraction of agents $\theta^{i,k}$ whose types are in the set $\{\theta \in \Theta : D^\theta(P) = c\}$. By the Vapnik-Chervonenkis Theorem,[4] there exists exists α_1 such that the probability

$$\Pr\{\sup_{P\in[0,1]^C} |D(P|\eta^k) - D(P|\eta)| > \epsilon/2\alpha_0\} \leq \alpha_1 \cdot \exp(-\frac{k}{8}(\frac{\epsilon}{2\alpha_0})^2).$$

Note that this bound is uniform in P.

If this is the case, by equation (8), the distance of all market clearing cutoffs P^k of F^k is bounded by

$$\begin{aligned} ||P^k - P^*|| &\leq \alpha_0\epsilon/2\alpha_0 + \alpha_0 \cdot ||S - S^k|| \\ &\leq \epsilon/2 + \alpha_0/k. \end{aligned}$$

Therefore, for $k \geq k_0 \equiv 2\alpha_0/\epsilon$,

$$\|P^k - P^*\| \leq \epsilon.$$

This implies that there exist $\alpha \geq 0$ and $\beta > 0$ such that the probability that F^k has any market clearing cutoffs with $|P^k - P^*| > \epsilon$ is lower than $\alpha e^{-\beta k}$, for any $k \geq k_0$. Moreover, we may take α such that the bound is only informative for $k \geq k_0$, so that the bound holds for all k as in the proposition statement. This completes the proof.

Part (3): Bound on G^k.

Let $\bar{f}$ be the supremum of the density of η. Denote the set of agents with scores which have at least one coordinate close to P^*_c as

$$\bar{\Theta} = \{\theta \in \Theta : \exists c \in C : \|e^\theta_c - P^*_c\| \leq \epsilon/4C\bar{f}\}.$$

[4] See Theorem 12.5 in Devroye et al. (1996) p. 197. As remarked in p. 198, the bound given in p. 197 is looser than the bound originally established by Vapnik and Chervonenkis (1971), which we use. The simple proof given in Devroye et al. (1996) follows the lines of Pollard (1984). The theorem can be proven using Hoeffding's inequality, and generalizes the Dvoretzky et al. (1956) inequality to the multidimensional case, and to arbitrary classes of measurable sets, not only sets of the form $\{x \in \mathbb{R}^n : x \leq \bar{x}\}$. The important requirement for the theorem to apply in our setting is that the Vapnik-Chervonenkis dimension of the class of sets $\{\theta \in \Theta : D^\theta(P) = c\}$ is finite.

The η measure of the set $\bar{\Theta}$ is bounded by

$$\eta(\bar{\Theta}) \leq 2C\bar{f} \cdot (\epsilon/4C\bar{f}) = \epsilon/2.$$

Let the agents in finite economy F^k be $\theta^{1,k}, \theta^{2,k}, \ldots, \theta^{k,k}$. The fraction of agents in economy F^k that have types in $\bar{\Theta}$ is given by the random variable

$$\tilde{G}^k = \sum_{i=1,\ldots,k} 1_{\theta^{i,k} \in \bar{\Theta}}/k.$$

By the Vapnik-Chervonenkis Theorem, in the argument of Part (2), we could have taken the constants $\alpha' \geq 0$ and $\beta' > 0$ in a way that the probability that both the fraction of agents with types in $\bar{\Theta}$ differs from the expected number $\eta(\bar{\Theta}) \leq \epsilon/2$ by more than $\epsilon/2$ is lower than $\alpha' e^{-\beta' k}/2$, and the probability

$$\Pr\{\sup_{P \in [0,1]^C} \|D(P|\eta^k) - D(P|\eta)\| > \epsilon/2\alpha_0\} \leq \alpha' e^{-\beta' k}/2.$$

If neither event happens, then $\tilde{G}^k \leq \epsilon/2 + \epsilon/2 = \epsilon$. Moreover, whenever this is the case all agents θ matched to a college different than $D^\theta(P^*)$ must be in $\bar{\Theta}$, so that $G^k \leq \tilde{G}^k \leq \epsilon$. The probability that neither event happens is at least $1 - \alpha' e^{-\beta' k}/2 - \alpha' e^{-\beta' k}/2 = 1 - \alpha' e^{-\beta' k}$. Therefore, the probability that $G^k > \epsilon$ is bounded above by $\alpha' e^{-\beta' k}$, as desired. □

C.4. **Non-Robustness.** Finally, we now prove the result on non-robustness of the set of stable matchings when there are multiple stable matchings.

Proof. (Proposition 1)

Suppose $P > P^-$; the case $P < P^+$ is analogous. Assume N is small enough such that all points $P' \in N$ satisfy $P' > P^-$. Denote $E = [\eta, S]$, and let $E^k = [\eta, S^k]$, where $S^k_c = S_c + 1/Ck$. Consider a sequence $\{P^k\}_{k \in \mathbb{N}}$ of market clearing cutoffs of E^k. Then

$$\sum_{c \in C} D_c(P^k|\eta) = \frac{1}{k} + \sum S_c.$$

Note that, for all points P' in N,

$$\sum_{c \in C} D_c(P'|\eta) \leq \sum_{c \in C} D_c(P^-|\eta) = \sum S_c < \sum S^k_c.$$

However, for large enough k, $\sum S^k_c < 1$, which means that for any market clearing cutoff P^k of E^k we must have $D(P^k|\eta) = S^k_c$, and therefore there are no market clearing cutoffs in N. □

Supplementary Appendix (Not for Publication)

The supplementary appendix contains additional results discussed in the paper. Appendix D extends the basic model to allow for flexible wages and contracts, albeit maintaining the assumption of responsive preferences. Appendix E discusses the connection between pre-matchings and cutoffs, and Appendix F gives an example of a school choice problem where DA-STB is inefficient.

Appendix D. Matching with Flexible Wages and Contracts

In many markets, agents negotiate not only who matches with whom, but also wages and other contractual terms. When hiring faculty most universities negotiate both in wages and teaching load. Firms that supply or demand a given production input may negotiate, besides the price, terms like quality or timeliness of the deliveries. This section extends the continuum model to include these possibilities. Remarkably, it is still the case that stable matchings have the simple cutoff structure described above. We highlight, however, that this holds under the assumption of responsive preferences, which we make throughout. The extension permits the comparison of different market institutions, such as personalized versus uniform wages.

D.1. **The Setting.** Following the standard terminology, we now consider a set of **doctor types** $\theta \in \Theta$ distributed according to a measure η, a finite set of **hospitals** $h \in H$, with H also denoting the number of hospitals, and a set of **contracts** X. η is defined over a σ-algebra Σ^Θ. Each contract x in X specifies

$$x = (\theta_x, h_x, w_x).$$

That is, a doctor θ_x, a hospital h_x, and other terms of the contract w_x. In addition, the set of contracts is assumed to contain an empty contract $\emptyset \in X$, which corresponds to being unmatched. A case of particular interest, which we later return to, is when w is a wage, and agents have quasilinear preferences.

A **matching** is a function $\mu : \Theta \cup H \to X \cup 2^X$, such that

(1) For all $\theta \in \Theta$: $\mu(\theta) \in \{x : x = \emptyset \text{ or } \theta_x = \theta\}$.
(2) For all $h \in H$: $\mu(h) \subseteq \{x : h_x = h\}$, the set $\{\theta_x : x \in \mu(h)\}$ is measurable, and $\eta(\{\theta_x : x \in \mu(h)\}) \leq S_h$.
(3) If $h_{\mu(\theta)} = h$ then $\mu(\theta) \in \mu(h)$, and if for some $x \in \mu(h)$ we have $\theta_x = \theta$, then $\mu(\theta) = x$.

That is, a matching associates each doctor (hospital) to a (set of) contract(s) that contains it, or to the empty contract. In addition, each doctor can be assigned to at most one contract. We say that a hospital h and a doctor θ are matched at μ if $h_{\mu(\theta)} = h$. Moreover, hospitals must be matched to a set of doctors of measure not exceeding its capacity S_h. Finally, (3) is a consistency condition that a doctor is matched to a hospital iff the hospital is matched to the doctor.

Models of matching with contracts have been proposed by Kelso and Crawford (1982); Hatfield and Milgrom (2005).[5] Those papers define stable matchings with respect to preferences of firms over sets of contracts. We focus on a simpler model, where stability is defined with respect to preferences of firms over single contracts. This corresponds to the approach that focuses on responsive preferences in the college admissions problem. This restriction considerably simplifies the exposition, as the same arguments used in the previous sections may be applied. Responsive preferences are considerably more restrictive than the most general preferences considered in the literature. In what follows, we assume that hospitals have **preferences** over single contracts that contain it, and the empty contract, and agents have preferences over contracts that contain them and over being unmatched.

Assume that doctors' preferences can be expressed by a **utility function** $u^\theta(x)$, and hospitals' by a utility function $\pi_h(x)$. The utility of being unmatched is normalized to 0. In the continuum model, we impose some restrictions on preferences and the set of available contracts. Let X_h^θ be the set of contracts that contain both a hospital h and a doctor θ.

Assumption D1. ***(Regularity Conditions)***

- **(Compactness)** There exists $M > 0$ such that, for any doctor-hospital pair θ, h, the set
$$\{(u^\theta(x), \pi_h(x)) | x \in X_h^\theta\}$$
is a compact subset of $[0, M]^2$.
- **(No Redundancy)** Given θ, h, no contract in $x \in X_h^\theta$ weakly Pareto dominates, nor has the same payoffs as another contract $x' \in X_h^\theta$.
- **(Completeness)** Given a hospital $h \in H$ and $0 < p \leq M$, there exists a doctor θ such that $X_{h'}^\theta = \emptyset$ for $h' \neq h$, and
$$\sup_{x:u^\theta(x)>0} \pi_h(x) = \max_{x:u^\theta(x)\geq 0} \pi_h(x) = p.$$
- **(Measurability)** Given any Lebesgue measurable set K in $\mathbb{R}^2$ and $h \in H$, the σ-algebra Σ^Θ contains all sets of the form
$$\{\theta \in \Theta | K = \{(u^\theta(x), \pi_h(x)) | x \in X_h^\theta\}\}.$$

[5]See Sönmez and Switzer (2012); Sönmez (2013) for applications of these models to real-life market design problems.

The assumptions accommodate many cases of interest, such as the transferable utility model considered below. The conditions make the analysis based on cutoffs tractable for the following reasons. The compactness and no redundancy assumptions guarantee that, for any hospital-doctor pair, there always exists a unique contract that is optimal for the doctor subject to giving the hospital a minimum level of profits. Completeness guarantees that there is always a marginal doctor type that is not matched to a hospital, even if this doctor type is not in the support of the distribution of types. The measurability condition guarantees that the measure η over the set of doctors is sufficiently rich so that a matching defined by cutoffs is measurable.

We are now ready to define stable matchings. Note that, since we only consider contracts yielding non-negative payoffs to both parties, we do not have to worry about individual rationality. A doctor-hospital pair θ, h is said to **block** a matching μ if there is a contract $x = (\theta, h, w)$ that θ prefers over $\mu(\theta)$, that is $u^\theta(x) > u^\theta(\mu(\theta))$, and either (i) hospital h does not fill its capacity $\eta(\{\theta_{x'} : x' \in \mu(h)\}) < S_h$, and h prefers x to the empty contract, $\pi_h(x) > 0$, or (ii) h is matched to a contract $x' \in \mu(h)$ which it likes strictly less than contract x, that is $\pi_h(x') < \pi_h(x)$.

Definition D1. A matching μ is **stable** if it has no blocking pairs.

D.2. **Cutoffs.** We now show that, within our matching with contracts framework, the allocation of doctors to hospitals is determined by an H-dimensional vector of cutoffs. It is convenient to think of cutoffs as the marginal value of capacity at each hospital – how much utility the hospital would gain from a small increase in capacity. Cutoffs are numbers $P_h \in [0, M]$, and a vector of cutoffs $P \in [0, M]^H$.

Denote an agent's maximum utility of working for a hospital h and providing the hospital with utility of at least P_h as

$$\begin{aligned} \bar{u}_h^\theta(P) &= \sup u^\theta(x) \\ \text{s.t. } & x \in X_h^\theta \\ & \pi_h(x) \geq P_h. \end{aligned}$$

We refer to this as the **reservation utility**[6] that hospital h offers doctor θ. Note that the reservation utility may be $-\infty$ if the feasible set $\{x \in X_h^\theta : \pi_x(x) \geq P_h\}$ is empty. Moreover, whenever this sup is finite, it is attained by some contract x, due to the compactness assumption. We define $\bar{u}_\emptyset^\theta(\cdot) \equiv 0$.

Now define a doctor's demand as the hospital that offers her the highest reservation utility given a vector of cutoffs. Note that doctors demand hospitals, and not contracts.

[6]Note that, even though formally $\bar{u}_h^\theta(P)$ depends on the entire vector P, it is constant on the coordinates P_{-h}.

The **demand** of a doctor θ given a vector of cutoffs P is

$$D^\theta(P) = \arg\max_{H\cup\{\emptyset\}} \bar{u}^\theta_h(P),$$

Demand may not be uniquely defined, as an agent may have the same reservation utility in more than one hospital.

Henceforth, we will make the following assumption, which guarantees that demand is uniquely defined for almost all doctors.

Assumption D2. ***(Strict Preferences)*** *For any cutoff vector $P \in [0, M]^C$, and hospital h, the following sets have η-measure 0:*

- The set of doctors with $\bar{u}^\theta_h(P) = \bar{u}^\theta_{h'}(P) > 0$ for some hospital $h' \neq h$.
- The set of doctors for which $\bar{u}^\theta_h(P) = 0$.
- The set of doctors for which $\bar{u}^\theta_h(P)$ is not continuous at P.

The first two requirements ask that, for any vector of cutoffs P, the set of doctors who are indifferent between the best offers of two hospitals, or of a hospital and being unmatched, has measure 0. This is true if there is sufficient heterogeneity of preferences in the population, with types having a non-atomic distribution, hence why this is termed a strict preferences assumption. The third condition is that, at a fixed P, reservation utility varies continuously for almost all doctors. The intuition is that, since reservation utility is decreasing in P_h, it can be discontinuous for at most a countable number of values of P_h. The assumption is that there is sufficient heterogeneity among doctors such that these discontinuities coincide for only a measure 0 set of doctors.

From now on, we fix a measurable selection from the demand correspondence, so that it is a function. The **aggregate demand** for a hospital is defined as

$$D_h(P) = \eta(\{\theta \in \Theta : D^\theta(P) = h\}).$$

The aggregate demand vector is defined as $D(P) = \{D_h(P)\}_{h\in H}$. Note that $D_h(P)$ does not depend on the arbitrarily defined demand of agents which are indifferent between more than one hospital, by the strict preferences assumption. Furthermore, demand is continuous in P, as shown by the following claim.

Claim D1. $D(P)$ is continuous in P

Proof. Take an arbitrary vector of cutoffs P_0 and constant $\epsilon > 0$. To establish continuity we will show that there there exists $\delta_\epsilon > 0$ such that $\|D(P) - D(P_0)\| < \epsilon$ for any P with $\|P - P_0\| \leq \delta_\epsilon$. To see this, define for any $\delta > 0$ the set

$$\begin{aligned}\Theta_\delta = \{\theta \in \Theta : \ & |\bar{u}^\theta_h(P) - \bar{u}^\theta_h(P_0)| < \epsilon/2, \text{ for all } h \text{ and } P \text{ with } \|P - P_0\| < \delta, \\ & |\bar{u}^\theta_{h'}(P_0) - \bar{u}^\theta_h(P_0)| > \delta, \text{ for all hospitals } h, h', \text{ and} \\ & |\bar{u}^\theta_\emptyset(P_0) - \bar{u}^\theta_h(P_0)| > \delta, \text{ for all hospitals } h\}.\end{aligned}$$

Note that the intersection of all such sets is contained in the following set:

$$\begin{aligned}\cap_{\delta>0}\Theta_\delta \subseteq \{\theta\in\Theta: \ & \bar{u}_h^\theta \text{ is continuous at } P_0 \text{for all } h,\\ & \bar{u}_{h'}^\theta(P_0)\neq\bar{u}_h^\theta(P_0), \text{ for all hospitals } h,h', \text{ and}\\ & \bar{u}_\emptyset^\theta(P_0)\neq\bar{u}_h^\theta(P_0), \text{ for all hospitals } h\}.\end{aligned}$$

Moreover, by Assumption D2, this latter set has measure 0. Since the Θ_δ are nested, we can take δ_ϵ small enough such that $\eta(\Theta_{\delta_\epsilon})<\epsilon/2$, and $\delta_\epsilon<\epsilon/2$.

To complete the proof, we will show that, for any P such that $\|P-P_0\|<\delta_\epsilon$, we have $\|D(P)-D(P_0)\|<\epsilon$. To see this, consider $\theta\in\Theta_{\delta_\epsilon}$. If $D^\theta(P_0)\neq\emptyset$, let $h=D^\theta(P_0)$. Then, for any $h'\neq h$,

$$\bar{u}_{h'}^\theta(P)<\bar{u}_{h'}^\theta(P_0)+\epsilon/2<\bar{u}_h^\theta(P_0)-\epsilon/2+\epsilon/2=\bar{u}_h^\theta(P).$$

The first inequality follows from the first condition in the definition of Θ_{δ_ϵ}. The second inequality follows from $\bar{u}_h^\theta(P_0)>\bar{u}_h^\theta(P)$, from the second condition in the definition of Θ_{δ_ϵ}, and the fact that $\delta_\epsilon<\epsilon/2$. This argument, and an analogous argument with $\emptyset$ instead of h', implies that $D^\theta(P)=D^\theta(P_0)$. Likewise, an analogous argument holds when $D^\theta(P_0)=\emptyset$. Therefore, for all $\theta\in\Theta_{\delta_\epsilon}$, we have $D^\theta(P)=D^\theta(P_0)$. Since $\eta(\Theta_{\delta_\epsilon})<\epsilon$, we have that $\|D(P)-D(P_0)\|<\epsilon$, as desired.

A **market clearing cutoff** is defined exactly as in Definition 2. Given a stable matching μ, let $P=\mathcal{P}\mu$ be given by

$$P_h=\inf\{\pi_h(x)|x\in\mu(h)\},$$

if $\eta(\mu(h))=S_h$ and $P_h=0$ otherwise. Given a market clearing cutoff P, we define $\mu=\mathcal{M}P$ as follows. Consider first a doctor θ. If $D^\theta(P)=\emptyset$, then θ is unmatched: $\mu(\theta)=\emptyset$. If $D^\theta(P)=h\in H$, then $\mu(\theta)$ is defined as the contract that gives the highest payoff to h conditional on θ not having a better offer elsewhere. Formally,

$$\begin{aligned}\mu(\theta) \quad &= \quad \arg\max_{x\in X_h^\theta}\pi_h(x)\\ &\text{s.t.} \quad u^\theta(x)\geq\bar{u}_{h'}^\theta(P) \text{ for all } h'\neq h,\end{aligned} \tag{9}$$

Note that $\mu(\theta)$ is uniquely defined, by the compactness and no redundancy assumptions. Since we defined $\mu(\theta)$ for all doctors, we can uniquely define it for each hospital as

$$\mu(h)=\{\mu(\theta):\theta\in\Theta \text{ and } h_{\mu(\theta)}=h\}.$$

$\square$

We have the following extension of the supply and demand lemma.

Lemma D1. ***(Supply and Demand Lemma with Contracts)*** *If μ is a stable matching, then $\mathcal{P}\mu$ is a market clearing cutoff, and if P is a market clearing cutoff then $\mathcal{M}P$ is a stable matching.*

Proof. **Part 1.** If μ is a stable matching, then $P = \mathcal{P}\mu$ is a market clearing cutoff.

We begin by proving a claim that will be used in the proof.

Claim D2. For almost all θ and $h \in H \cup \{\phi\}$ such that $\mu(\theta) = h$, we have $D^\theta(P) = h$.

Proof. Assume, to reach a contradiction, that $D^\theta(P) = h' \neq h$ for a positive measure of doctors. By the definition of demand, we have that, for any such doctor θ, $\bar{u}^\theta_{h'}(P) \geq \max_{h'' \neq h'} \bar{u}^\theta_{h''}(P)$. Moreover, from the strict preferences assumption, there exists a positive mass of doctor types θ such that $\bar{u}^\theta_{h'}(P) > \max_{h'' \neq h'} \bar{u}^\theta_{h''}(P)$, and the functions $\bar{u}^\theta_{h''}(P)$ for each $h'' \in H \cup \{\emptyset\}$ are continuous at P. Let θ_0 be one such doctor. By the definition of $\bar{u}^{\theta_0}_c$ we have that $u^{\theta_0}(\mu(\theta_0)) \leq \bar{u}^{\theta_0}_h(P)$. Consequently, there exists a contract $x \in X^{\theta_0}_{h'}$ such that

$$\begin{aligned} u^{\theta_0}(x) &> u^{\theta_0}(\mu(\theta_0)) \\ \pi_{h'}(x) &> P_{h'}. \end{aligned}$$

We now show that this implies that θ_0 and h' block the matching μ. By definition of $\mathcal{P}$, and the completeness assumption, there exist contracts in $\mu(h')$ giving hospital h' payoffs arbitrarily close to $P_{h'}$. Therefore, there exists a contract $x' \in \mu(h')$ with $\pi_{h'}(x') < \pi_{h'}(x)$, so that h' and θ_0 block μ. This contradicts the fact that μ is stable. $\square$

This claim implies that $D(P) \leq S$. To prove that P is a market clearing cutoff, we only have to show that for any h such that $P_h > 0$, we have $D(P) = S$. To see this, note that, by the completeness assumption, there exists a doctor θ who may only contract with hospital h, and such that

$$\max_{x : u^\theta(x) \geq 0} \pi_h(x) < P_h,$$

and there exists a contract $x \in X^\theta_h$ with

$$\begin{aligned} \pi_h(x) &> 0 \\ u^\theta(x) &> 0. \end{aligned}$$

Note that, by the definition of $\mathcal{P}$, θ is not matched to h at μ, as h is only matched to contracts that yield utility of at least P_h. Therefore, θ is unmatched at μ, that is $\mu(\theta) = \emptyset$. Since μ is stable, θ and h cannot be a blocking pair, and therefore h must be matched to a mass S_h of doctors, that is

$$\eta(\{\theta : h_{\mu(\theta)} = h\}) = S_h.$$

Using the claim proved above, we have that

$$D_h(P) = \eta(\{\theta : h_{\mu(\theta)} = h\}) = S_h,$$

completing the proof of Part 1.

Part 2. If P is a vector of market clearing cutoffs, then $\mu = \mathcal{M}P$ is a stable matching.

First note that μ is a matching. It satisfies the requirement that doctors are matched to hospitals, that hospitals are matched to sets of doctors, and the consistency requirements by definition. Since $D^\theta(P)$ is a measurable selection from the demand correspondence, it satisfies the requirement that hospitals are matched to measurable sets of doctors. And finally, it satisfies that each hospital is matched to a set of doctors not exceeding its capacity, because P being a market clearing cutoff implies $D(P) \leq S$.

We now show that μ is stable. The proof uses the following claim.

Claim D3. For all $h \in H$ and $x \in \mu(h)$, we have $\pi_h(x) \geq P_h$.

Proof. Let $\theta = \theta_x$. By definition of $\mathcal{M}$,

$$\bar{u}^\theta_h(P) \geq \max_{h' \in H \cup \{\emptyset\}} \bar{u}^\theta_{h'}(P).$$

Therefore, by definition of $\bar{u}^\theta_h$, there exists $x' \in X^\theta_h$ such that

$$\begin{aligned} u^\theta(x') &\geq \max_{h' \in H \cup \{\emptyset\}} \bar{u}^\theta_{h'}(P) \\ \pi_h(x') &\geq P_h. \end{aligned}$$

Moreover, by the definition of $\mathcal{M}$, we have $\pi_h(x) \geq \pi_h(x')$. Consequently, $\pi_h(x) \geq P_h$, as desired. □

To see that μ is stable, note that, by the no redundancy assumptions, no contracts are Pareto dominated, so that there can only be blocking pairs formed of agents who are not matched to each other. Consider a pair θ, h, who are not matched at μ. We will show they cannot form a blocking pair. First note that, by the definition of $\mathcal{M}$,

$$u^\theta(\mu(\theta)) \geq \bar{u}^\theta_h(P). \tag{10}$$

Consider now the case where $D_h(P) < S_h$. Therefore, $P_h = 0$. This implies that

$$\bar{u}^\theta_h(P) = \max_{x \in X^\theta_h} u^\theta(x).$$

Equation (10) then implies that

$$u^\theta(\mu(\theta)) \geq \max_{x \in X^\theta_h} u^\theta(x),$$

and therefore θ and h are not a blocking pair.

Finally consider the case where $D_h(P) = S_h$. The definition of $\mathcal{M}$ then implies that the mass of doctors matched to h at μ equals S_c. By Claim D3, for all contracts

$x \in \mu(h)$, we have $\pi_h(x) \geq P_h$. If there exists $x' \in X_h^\theta$ such that $\pi_h(x') \geq P_h$, we then have that $u^\theta(x') \leq \bar{u}_h^\theta(P)$. Therefore, by equation (10), we have that $u^\theta(x') \leq u^\theta(\mu(\theta))$. Consequently, θ and h are not a blocking pair. This completes the proof. □

Note that, in the matching with contracts setting, there is no longer a bijection between market clearing cutoffs and stable matchings. This happens for two reasons, one substantial and one technical. The substantial reason is that the contract terms w are not uniquely determined by cutoffs, as there is room for doctors and hospitals to share the surplus of relationships in different ways, without violating stability. The technical reason is that we have not imposed a condition akin to the open on the right condition in the model from Section 2, which precludes multiplicities of stable matchings that differ in measure 0 sets.

D.3. **Existence.** To establish the existence of a stable matching, we must modify the previous argument, which used the deferred acceptance algorithm. One simple modification is using a version of the algorithm that Biró (2007) terms a "score limit algorithm", which calculates a stable matching by progressively increasing cutoffs to clear the market. A straightforward application of Tarski's fixed point theorem gives us existence in this case.

Proposition D1. *A stable matching with contracts always exists.*

Proof. Consider the operator $T : [0, M]^H \to [0, M]^H$ defined by $P' = TP$ is the smallest solution $P' \in [0, M]^H$ to the system of inequalities

$$D_h(P'_h, P_{-h}) \leq S_h.$$

We will show that this operator has a fixed point, and that this fixed point is a market clearing cutoff.[7]

First note that, by the continuity of D_h, and since $D_h(M, P_h) = 0$, the smallest solution to this equation is well-defined. Therefore, T is well-defined. Moreover, since $D_h(P'_h, P_{-h})$ is weakly increasing in P_{-h} and weakly decreasing in P'_h we have that T is weakly increasing in P. We know that T takes the cube $[0, M]^H$ into itself, by definition. By Tarski's fixed point theorem, T has a fixed point.

It only remains to show that every fixed point P^* of T is a market clearing cutoff. By definition of T we have that $D_h(P^*) \leq S_h$ for all hospitals $h \in H$, so that demand for no hospital exceeds supply. Consider h such that $P_h^* > 0$. By definition of T we have that

$$D_h(P_h, P_{-h}^*) > S_h$$

[7]A cutoff limit algorithm can be described as starting with cutoffs of 0, and successively applying the operator T. What T does in each step is raising the cutoff of each hospital just enough to clear the market for the hospital given the cutoffs of other hospitals.

for any $P_h < P_h^*$. By continuity of demand we have that $D_h(P^*) \geq S_h$, which combined with the fact that demand does not exceed supply implies that $D_h(P^*) = S_h$, and therefore P^* is a market clearing cutoff. □

D.4. **The Quasilinear Case.** A particularly interesting case of the model is when contracts only specify a wage w, and preferences are quasilinear. That is, the utility of a contract $x = (\theta, h, w)$ is

$$\begin{aligned} u^\theta(x) &= u_h^\theta + w \\ \pi_h(x) &= \pi_h^\theta - w \end{aligned}$$

and contracts include all possible values of w, such that these values are in $[0, M]$. Define the **surplus** of a doctor-hospital pair as

$$s_h^\theta = u_h^\theta + \pi_h^\theta.$$

We assume that M is large enough so that, for all θ in the support of η we have $0 \leq s_i^\theta \leq M$, so that doctors and hospitals may freely divide the surplus of a relationship. We assume moreover that assumptions D1 and D2 hold. Denote a model satisfying the above properties as a matching with contracts model with quasilinear preferences. From the definition of reservation utility we get that, for all doctors in the support of η,

$$\bar{u}_h^\theta(P) = s_h^\theta - P_h.$$

Therefore, in any stable matching, doctors are sorted into the hospitals where $s_h^\theta - P_h$ is the highest, subject to it being positive. One immediate consequence is that doctors *do not go* necessarily to the hospital where they generate the largest surplus s_h^θ. If $P_h \neq P_{h'}$, it may be the case that $s_h^\theta > s_{h'}^\theta$, but doctor θ is assigned to h'. However, the allocation of doctors to hospitals does maximize the total surplus generated in the economy, given the capacity constraints (see Azevedo (2014) Appendix A.2). Figure D1 plots a stable matching in an economy with two hospitals.

Let the distribution of surplus vectors s^θ be η_s. We then have the following uniqueness result.

Proposition D2. *Consider a matching with contracts model with quasilinear preferences. If η_S has full support over $[0, M]^C$ then there is a unique vector of market clearing cutoffs.*

Proof. We begin by showing that the set of market clearing cutoffs is a lattice.

Claim D4. The set of market clearing cutoffs is a complete lattice.

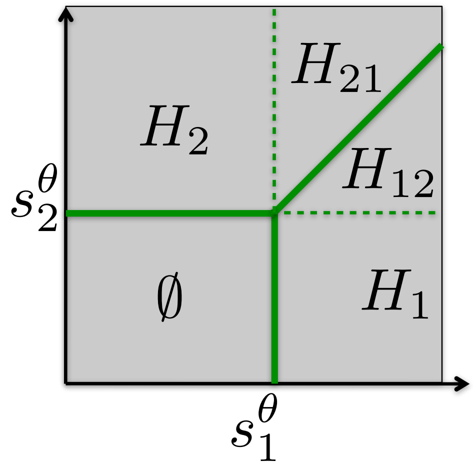

FIGURE D1. A matching with transferable utility with two hospitals. The square represents the set of possible surplus vectors s^θ. Doctors in regions H_1 and H_{12} are matched to hospital 1, and doctors in regions H_2 and H_{21} to hospital 2.

Proof. Define the operator $T : [0, M]^H \to [0, M]^H$ as follows. Let $TP = P'$, with P'_h being the solution to

$$D_h(P'_h, P_{-h}) = S_h \tag{11}$$

if such a solution $P'_h \in [0, M]$ exists, and 0 otherwise.[8] We will show that set of fixed points of T equals the set of market clearing cutoffs.

Note that, since $D_h(M, P_{-h}) = 0$ and demand is continuous, if a solution p to $D(p, P_{-h})$ does not exist, then $D(p, P_{-h}) < S_h$ for all $p \in [0, M]$, and therefore $P'_h = 0$ and $D(P'_h, P_{-h}) < S_h$. We will use this observation to show that the fixed points of T correspond to market clearing cutoffs.

Consider a fixed point P^* of T. For a given h, since $TP^* = P^*$, either equation (11) has a solution, and we have $D_h(P^*) = S_h$, or the equation has no solutions, in which case $D_h(P^*) < S_h$ and $P^*_h = 0$. Therefore, P^* is a market clearing cutoff.

Consider now a market clearing cutoff P^*. For any hospital $h \in H$, if $P^*_h > 0$, we have that $D_h(P^*) = S_h$, so that $(TP^*)_h = P^*_h$. If $P^*_h = 0$ we have that either the market clears exactly, $D_h(P^*) = S_h$, in which case $(TP^*)_h = P^*_h$, or that h is in excess supply, $S_h > D_h(P^*) \geq D_h(p, P^*_{-h})$ for all $p \in [0, M]$, and therefore $(TP^*)_h = 0 = P^*_h$. Since this holds for all hospitals, P^* is a fixed point.

Now that we have established that the set of fixed points of T equals the set of market clearing cutoffs, we can show that this set if a lattice. To see this, note that T is weakly

[8]Note that, if the solution to the equation defining P'_h exists, it is unique, as the full support assumption implies that the left side is strictly decreasing in P'_h.

increasing in P, and takes $[0, M]$ in itself. Therefore, by Tarski's Theorem, the set of fixed points is a non-empty complete lattice. $\square$

Let P^- and P^+ be the smallest and largest market clearing cutoffs. Let H^+ be the subset of hospitals for which $P_h^+ > P_h^-$. That is

$$H^+ = \{h \in H : P_h^+ > P_h^-\}.$$

In particular, for all $h \in H^+$ we have $P_h^+ > 0$.

If H^+ is empty, then $P^- = P^+$, and we are done. Assume henceforth that H^+ is nonempty. Since both P^- and P^+ are market clearing cutoffs, we have that

$$\sum_{h \in H^+} D_h(P^-) \leq \sum_{h \in H^+} S_h = \sum_{h \in H^+} D_h(P^+).$$

However, since $P_h^- = P_h^+$ for $h \notin H^+$, and $P_h^- < P_h^+$ for all $h \in H^+$ we have that

$$\sum_{h \in H^+} D_h(P^-) \geq \sum_{h \in H^+} D_h(P^+).$$

Therefore,

$$\sum_{h \in H^+} D_h(P^-) = \sum_{h \in H^+} D_h(P^+).$$

Under the assumption that the support of η_s is the set $[0, M]^H$, this can only be true if $P^- = P^+$, completing the proof. $\square$

The proposition guarantees that the allocation of doctors to hospital is unique, up to a measure 0 set of doctors.[9] However, the stable matching is not unique, as wages are not uniquely determined by stability. The intuition is that, in a stable matching, a hospital may offer a doctor any wage such that the doctor's utility is above that in her next best choice, and the hospital's gain from the relationship above its reservation value of capacity. Therefore, in general the surplus s_h^θ can be divided in different ways without compromising stability.

Appendix E. Relationship to Pre-Matchings

This section clarifies the connection and differences between the cutoffs approach to stable matchings, and the approach using pre-matchings proposed by Adachi (2000) and Echenique and Oviedo (2004, 2006).

The key advantage of the pre-matchings approach is that stable matchings can be analyzed as fixed points of a monotone operator in the set of pre-matchings. Although

[9]This result is related to a standard uniqueness result in markets with a finite number of participants. With a finite number of doctors and hospitals, the allocation of doctors to hospitals is generically unique. This is true because generically there is a unique allocation that maximizes aggregate surplus. In contrast, with a continuum of doctors, strict preferences, and full support of η_C, the matching that maximizes aggregate surplus is unique for almost every doctor.

the set of pre-matchings is much larger set than the set of matchings, this result is useful for deriving general lattice theoretic results, due to the monotonicity of the operator. Echenique and Oviedo (2006) use the fixed point result to establish results on the existence and structure of the set of stable matchings in models with very general preferences and many to many matching. The cutoff approach is different. One way to see this is that the set of cutoffs is much smaller than the set of all pre-matchings. Cutoffs are useful in different applications, such as deriving simple comparative statics with responsive preferences.

In this section we show that there is a connection between the two approaches. When colleges have responsive preferences, the intermediate steps of the monotone operator used by Echenique and Oviedo (2006) can be written in terms of cutoffs. With this observation, we can show that, in the case of responsive preferences, our result that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ restricted to the set of stable matchings is the identity is closely related to the result that fixed points of the monotone operator proposed by Adachi (2000) and Echenique and Oviedo (2006) correspond to stable matchings.

E.1. **Formal Definition of Pre-matchings and the Fixed Point Operator.** Consider a finite set C of colleges c and finite set $\tilde{\Theta}$ of students θ .

Definition E1. A **prematching** $\nu = (\nu^C, \nu^{\tilde{\Theta}})$ is a pair of functions

$$\begin{aligned} \nu^C : C &\longrightarrow 2^{\tilde{\Theta}} \\ \nu^{\tilde{\Theta}} : \tilde{\Theta} &\longrightarrow C \cup \tilde{\Theta}, \end{aligned}$$

such that, for all $\theta \in \tilde{\Theta}$, $\nu^{\tilde{\Theta}}(\theta) \in C$ or $\nu^{\tilde{\Theta}}(\theta) = \theta$.

That is, a prematching ν specifies a set of students $\nu^C(c)$ matched to each college c, and a college or remaining unmatched $\nu^{\tilde{\Theta}}(\theta)$ to each student θ. There is no consistency requirement between ν^C and $\nu^{\tilde{\Theta}}$. We denote the set of pre-matchings by $\mathcal{V} = \mathcal{V}^C \times \mathcal{V}^{\tilde{\Theta}}$, where $\mathcal{V}^C$ and $\mathcal{V}^{\tilde{\Theta}}$ are the set of functions that may be the first and second coordinates of a prematching. The set of pre-matchings can be thought of as being a larger set than the set of matchings because either coordinate has sufficient information to fully specify a matching.

We now follow Echenique and Oviedo (2006) in defining the operator T over the set of pre-matchings, with fixed points corresponding to stable matchings. The operator is defined in terms of choice functions, which they define using preference relations of agents over sets of match partners. For each college c we consider a **choice function** with, for any $A \subseteq \tilde{\Theta}$, $Ch(c, A)$ denoting the college's preferred subset of A. Likewise, for each student we consider a **choice function** $Ch(\theta, A)$ which picks a college in A or

θ, out of any subset $A \subseteq C$. Given a prematching ν, define

$$\begin{aligned} U(c,\nu) &= \{\theta \in \tilde{\Theta} : c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})\}, \text{ and} \\ V(\theta,\nu) &= \{c \in C : \theta \in Ch(c, \nu^{C}(c) \cup \{\theta\})\}. \end{aligned}$$

That is, a student is in $U(c,\nu)$ if he would choose c over his assignment under $\nu^{\tilde{\Theta}}$, and a college is in $V(\theta,\nu)$ if θ would be one of its chosen students out of $\nu^{C}(c) \cup \{\theta\}$. The **fixed point operator** $T : \mathcal{V} \to \mathcal{V}$ is is defined by

$$\begin{aligned} (T\nu)(\theta) &= Ch(\theta, V(\theta,\nu)) \\ (T\nu)(c) &= Ch(c, U(c,\nu)). \end{aligned}$$

E.2. **Definition of the Fixed Point Operator with Cutoffs in the Responsive Preferences Case.** We now return to the model of Section 3.1, where students have preferences $\succ^{\theta}$ over colleges, and colleges have preferences over individual students (given by e_c), and a capacity S_c.[10]

In this setting, the choice maps have a simple definition. Namely, if $|A| \leq S_c$ then $Ch(c,A) = A$ and if $|A| > S_c$ then $Ch(c,A)$ the subset of the S_c students in A with the highest scores. Likewise, $Ch(\theta,A)$ is student θ's preferred college in A if A is nonempty, or θ otherwise.

To define the operator T using cutoffs, we must define cutoffs on both sides of the market. A vector of cutoffs on the college side, denoted P^C, is defined as in the main text, and the set of all such cutoffs is $\mathbb{P}^C \equiv [0,1]^C$. A vector of cutoff in the student side is denoted $P^{\tilde{\Theta}}$. For each student θ, the cutoff $P^{\tilde{\Theta}}_{\theta} \in C$ or $P^{\tilde{\Theta}}_{\theta} = \theta$. Intuitively, the cutoff $P^{\tilde{\Theta}}_{\theta}$ denotes the least preferred college that student θ is willing to match with. The set of student side cutoffs is denoted as $\mathbb{P}^{\tilde{\Theta}} \equiv \times_{\tilde{\Theta}}(C \cup \{\theta\})$.

We now define the following operators. The first two extend the operators $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{M}}$ to pre-matchings, and the last two are the mirror images of $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{M}}$ on the other side of the market.

- Define $\mathcal{P}^C : \mathcal{V}^C \longrightarrow \mathbb{P}^C$ as, for each college c, the S_c-highest score of a student in $\nu^C(c)$, denoted as $\left(\mathcal{P}^C(\nu^C)\right)_c$, with the convention the the S_c-highest score is 0 when there are less than S_c students in $\nu^C(c)$. Notice that $\mathcal{P}^C$ coincides with $\tilde{\mathcal{P}}$ when ν^C corresponds to a stable matching.[11]
- Define $\mathcal{M}^{\tilde{\Theta}} : \mathbb{P}^C \longrightarrow \mathcal{V}^{\tilde{\Theta}}$ in similar fashion as $\tilde{\mathcal{M}}$ was defined in the main text,

$$(\mathcal{M}^{\tilde{\Theta}}(P^C))(\theta) = D^{\theta}(P^C).$$

[10] The college preferences over individual students can be extended to responsive preferences over sets of students.

[11] Formally, $\mathcal{P}^C(\nu^C) = \tilde{\mathcal{P}}\tilde{\mu}$ when for all $c \in C$ we have $\mu(c) = \nu^C(c)$ and $\#\nu^C(c) \leq S_c$.

- Define $\mathcal{P}^{\tilde{\Theta}} : \mathcal{V}^{\tilde{\Theta}} \longrightarrow \mathbb{P}^{\tilde{\Theta}}$ as
$$(\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta = \nu^{\tilde{\Theta}}(\theta).$$
- Define $\mathcal{M}^C : \mathbb{P}^{\tilde{\Theta}} \longrightarrow \mathcal{V}^C$ as
$$(\mathcal{M}^C(P^{\tilde{\Theta}}))(c) = Ch(c, \{\theta : c \succeq^\theta P^{\tilde{\Theta}}(\theta)\}).$$

The following Claim clarifies how the map T can be written in terms of cutoffs.

Claim E1. For any prematching $\nu \in \mathcal{V}$ we have
$$T\nu = (\mathcal{M}^C \mathcal{P}^{\tilde{\Theta}} \nu^{\tilde{\Theta}}, \mathcal{M}^{\tilde{\Theta}} \mathcal{P}^C \nu^C).$$

Proof. We begin with the student side. Consider a student θ. Notice that, by definition of the set $V(\theta, \nu)$ we have
$$V(\theta, \nu) = \{c \in C : \theta \in Ch(c, \nu^C(c) \cup \{\theta\})\}.$$
Moreover, by the definition of the choice function, θ is chosen by college c if and only if e^θ_c is at least as high as the S_c-highest score at college c among the students in $\nu^C(c)$. Therefore, we have
$$V(\theta, \nu) = \{c \in C : \ e^\theta_c \geq \left(\mathcal{P}^C(\nu^C)\right)_c\}.$$

Using this equation, we can write the student side of the operator T as
$$\begin{aligned}(T\nu)(\theta) &= Ch(\theta, V(\theta, \nu)) \\ &= Ch(\theta, \{c \in C : \ e^\theta_c \geq \left(\mathcal{P}^C(\nu^C)\right)_c\}) \\ &= D^\theta(\mathcal{P}^C(\nu^C)).\end{aligned}$$
By the definition of $\mathcal{M}^{\tilde{\Theta}}$, this equals $\mathcal{M}^{\tilde{\Theta}} \mathcal{P}^C(\nu^C)$ as desired.

Consider now the college side. Fix a college c. By definition of the set $U(c, \nu)$ we have
$$U(c, \nu) = \{\theta \in \tilde{\Theta} : c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})\}.$$
Note that $c \in Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})$ if and only if $c = Ch(\theta, \nu^{\tilde{\Theta}}(\theta) \cup \{c\})$, as students match to at most a single college. This is the case if and only if $c \succeq^\theta \nu^{\tilde{\Theta}}(\theta)$. Moreover, by definition of $\mathcal{P}^{\tilde{\Theta}}$ we have $(\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta = \nu^{\tilde{\Theta}}(\theta)$. Therefore,
$$U(c, \nu) = \{\theta \in \tilde{\Theta} : c \succeq^\theta (\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta\}.$$
Consider now the operator T. We have
$$\begin{aligned}(T\nu)(c) &= Ch(c, U(c, \nu)) \\ &= Ch(c, \{\theta \in \tilde{\Theta} : c \succeq^\theta (\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}}))_\theta\}).\end{aligned}$$
By the definition of the operator $\mathcal{M}^C$, we have
$$(T\nu)(c) = (\mathcal{M}^C(\mathcal{P}^{\tilde{\Theta}}(\nu^{\tilde{\Theta}})))(c),$$

completing the proof. □

We now discuss the relationship between our results and those of Echenique and Oviedo (2004, 2006) and Adachi (2000). We say that a matching μ is **associated** with a prematching $g(\mu) = \nu$ by letting each coordinate be the matching restricted to colleges or students, i.e., $\nu = (\mu|_C, \mu|_{\tilde{\Theta}})$. Note that the function g is injective. If $g^{-1}(\nu) \neq \emptyset$ we say that prematching ν is **associated** with matching $g^{-1}(\nu)$. Adachi's main result is that a prematching v is a fixed point of T iff it is associated with a stable matching.

First note that Adachi result is distinct from the fact that stable matchings are associated with market clearing cutoffs. Since cutoffs are only an intermediate step in the operator T, the fact that the fixed points of T correspond to stable matchings is distinct from the relationship between market clearing cutoffs and stable matchings.

However, the fixed point result is closely related to our result that $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ restricted to stable matchings is the identity map. With an analogous argument to the one we used to prove that result, it is possible to prove that, given a stable matching μ, the operator $\mathcal{M}^C\mathcal{P}^{\tilde{\Theta}}$ (or $\mathcal{M}^{\tilde{\Theta}}\mathcal{P}^C$) takes $\mu|_{\tilde{\Theta}}$ into $\mu|_C$ (or $\mu|_C$ into $\mu|_{\tilde{\Theta}}$). By Claim E1, this is equivalent to Adachi's result that pre-matchings corresponding to stable matchings are fixed points of T. Therefore, we could have derived Adachi's fixed point result using cutoffs, or proven the result about $\tilde{\mathcal{M}}\tilde{\mathcal{P}}$ using Adachi's result. Note however that this argument is only valid with responsive preferences, so that we cannot use this argument to establish the more general results of Echenique and Oviedo (2004, 2006).

Appendix F. A Large Market Where Deferred Acceptance with Single Tie-Breaking Is Inefficient

This section presents a simple example that shows that the deferred acceptance with single tie-breaking mechanism (see Section 4.3) can produce Pareto dominated outcomes for a large share of the students with high probability, even in a large market.[12]

Example F1. (School Choice)

A city has two schools $c = 1, 2$ with the same capacity. Students have priorities to schools according to the walk zones where they live in. Half of the students live in the walk zone of each school. In this example, the grass is always greener on the other side, so that students always prefer the school to which they don't have priority. The city uses the DA-STB mechanism. To break ties, the city gives each student a single lottery number l uniformly distributed in $[0, 1]$. The student's score is

$$l + I(\theta \text{ is in } c\text{'s walk zone}).$$

[12]The example is a continuum version of an example used by Erdil and Ergin (2008) to show a shortcoming of deferred acceptance with single tie-breaking: it may produce matchings which are ex post inefficient with respect to the true preferences, before the tie-breaking, being dominated by other stable matchings.

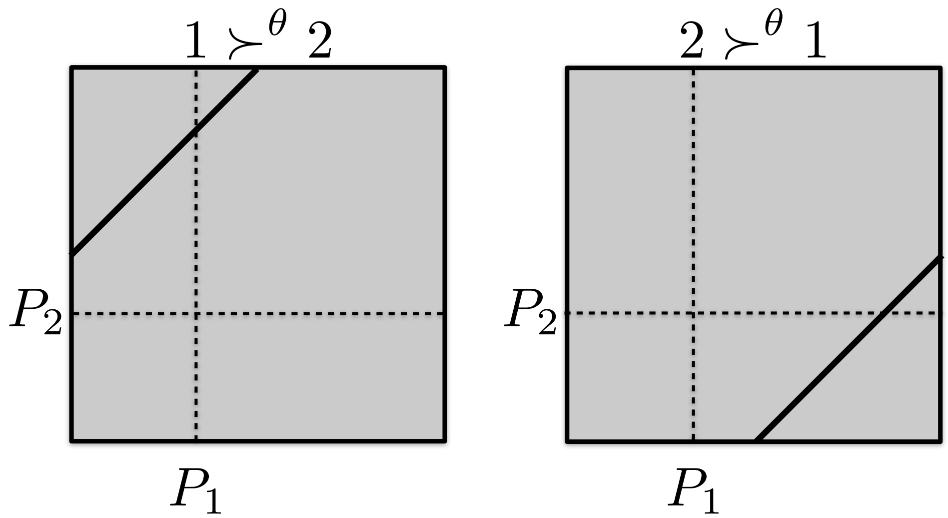

Figure F1. The distribution of student types in Example F1. The unit mass of students is uniformly distributed over the solid lines. The left square represents students in the walk zone of school 2, and the right square students with priority to school 1. The dashed lines represents one of an infinite number of possible vectors of market clearing cutoffs $P_1 = P_2$.

In the continuum economy induced by the DA-STB mechanism, there is a mass 1/2 of students in the walk zone of each school, and $S_1 = S_2 = 1/2$. Figure F1 depicts the distribution of students in the economy.

We now analyze the stable matches in this continuum economy. Note that market clearing cutoffs must be in $[0,1]$, as the mass of students with priority to each school is only large enough to exactly fill each school. Consequently, the market clearing equations can be written, for $0 \leq P \leq (1,1)$, as

$$\begin{aligned} 1 &= 2 \quad S_1 = (1 - P_1) + P_2 \\ 1 &= 2 \quad S_2 = (1 - P_2) + P_1. \end{aligned}$$

The first equation describes demand for school 1. $1 - P_1/2$ students in the walk zone of 2 are able to afford it, and that is the first term. Moreover, an additional $P_2/2$ students in the walk zone of 1 would rather go to 2, but don't have high enough lottery number, so they have to stay in school 1. The market clearing equation for school 2 is the same.

These equations are equivalent to

$$P_1 = P_2.$$

Hence, any point in the line $\{P = (x,x) | x \in [0,1]\}$ is a market clearing cutoff - the lattice of stable matchings has infinite points, ranging from a student-optimal stable matching, $P = (0,0)$ to a school-optimal stable matching $P = (1,1)$.

Now consider a slightly different continuum economy, that has a small mass of agents that have no priority, so that the new mass has e^θ uniformly distributed in $[(0,0),(1,1)]$. It's easy to see that in that case the unique stable matching is $P=(1,1)$. Therefore, adding this small mass undoes all stable matchings of the original continuum economy, except for $P=(1,1)$. In addition, it is also possible to find perturbations that undo the school-optimal stable matching $P=(1,1)$. If we add a small amount ϵ of capacity to school 1, the unique stable matching is $P=(0,0)$. And if we reduce the capacity of school 1 by $\epsilon/2$, the unique stable matching is $P=(1+\epsilon,1)$, which is close to $P=(1,1)$.

Consider now a school choice problem where, in addition to these students, there is a small mass of students that have no priority, and prefer school 1. By the argument in the above paragraph, the only stable matching of the continuum economy induced by the new school choice problem is one where $P=(1,1)$. Therefore, this school choice problem leads to agents attending the school to which they have priority with probability close to 1. However, it is a stable allocation for all students with priorities to attend the school they prefer. Therefore, DA-STB produces outcomes that are Pareto dominated for many students relative to a stable allocation. This is in contrast with the result by Che and Kojima (2010), who show that in the case without priorities the RSD mechanism is approximately ordinally efficient in a large market.

Appendix G. Asymptotic Distribution of Cutoffs

This section derives the asymptotic distribution of cutoffs in randomly generated discrete economies as in Section 4.2. The results show that the asymptotic distribution is a multivariate normal, with mean centered at the continuum cutoffs, covariance matrix given by a formula below, and standard deviations proportional to the inverse of the square root of the number of students. We then gauge the fit of the asymptotic distribution to realistic market sizes with simulations. In what follows $\xrightarrow{d}$ denotes convergence in distribution. The following result uses similar assumptions as Proposition 3.

Proposition G1. *Assume that the continuum economy $E=[\eta,S]$ admits a unique market clearing cutoff P^*, and $\sum_c S_c<1$. Let $F^k=[\eta^k,S^k]$ be a randomly drawn finite economy, with k students drawn independently according to η and the vector of capacity per student S^k defined as $S^k k=[Sk]$. Let $\{P^k\}_{k\in\mathbb{N}}$ be a sequence of random variables, such that each P^k is a market clearing cutoff of F^k. Assume that E has a C^1 demand function, and that $\partial D(P^*)$ nonsingular. Then the asymptotic distribution of the difference between P^k and P^* satisfies*

$$\sqrt{k}\cdot(P^k-P^*)\xrightarrow{d}\mathcal{N}(0,\partial D(P^*)^{-1}\cdot\Sigma^D\cdot(\partial D(P^*)^{-1})'),$$

where $\mathcal{N}(\cdot,\cdot)$ denotes a C-dimensional normal distribution with given mean and covariance matrix. The matrix Σ^D is equal to

$$\Sigma^D = \begin{pmatrix} S_1(1-S_1) & -S_1S_2 & \cdots & \cdots & -S_1S_C \\ -S_2S_1 & S_2(1-S_2) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & -S_{C-1}S_C \\ -S_CS_1 & \cdots & \cdots & -S_CS_{C-1} & S_C(1-S_C) \end{pmatrix}.$$

Before proving the proposition, we report the results of simulations testing the asymptotic approximation. We randomly generated discrete economies with the distribution of preferences from the example in Section 2.3. For each market size we performed 10,000 simulations. For each simulation we calculated a cutoff P^k corresponding to the student-optimal stable matching.

Figure G1 displays the joint distribution of market clearing cutoffs across simulations. As predicted, when the number of students increases, the distribution is increasingly concentrated around the limit market clearing cutoffs. Moreover, convergence is fast. For instance, with 50 students per college the distribution is already highly concentrated around P^*. Consistent with the simulations in Section 4.2, these results reinforce that the continuum model is a good approximation in realistically sized markets with many students per college.

To test the finer predictions of the asymptotic approximation, Figure G2 plots the distribution of normalized market clearing cutoffs, $\sqrt{k}(P^k - P^*)$. Proposition G1 predicts that this joint distribution converges to a normal distribution with zero mean and covariance matrix given by

$$\partial D(P^*)^{-1} \cdot \Sigma^D \cdot (\partial D(P^*)^{-1})' \approx \begin{pmatrix} 0.3665 & -0.0000 \\ -0.0000 & 0.3210 \end{pmatrix}. \tag{12}$$

We find that, for as few as 10 students per college, the distributions are quite close, with very similar means and covariance matrices. To illustrate this, the bottom panels plot histograms of $\sqrt{k}(P_1^k - P_1^*)$ in the simulations. The panels also plot the theoretical normal distribution, predicted according to Proposition G1 and the covariance formula in equation (12). The predicted distribution matches the actual histograms very closely. We note that, although the means, covariances, and overall shapes of the actual and theoretical distributions are quite close, the histograms clearly show deviations from the normal distribution. For example, the histogram with 50 students per college displays some skew, unlike the normal distribution. Therefore, for small values of k, the asymptotic distribution has a good overall fit, but may fail to match higher moments accurately.

We end this section with a proof of the proposition. The proof follows standard asymptotic derivations in statistics, using a delta method argument. The only additional difficulty is that the discrete excess demand curve $z(\cdot|\eta^k)$ is not differentiable. We circumvent this issue with Claim G1 below, which shows that it is possible to follow the delta method argument by moving along the graph of $z(\cdot|\eta)$ instead of $z(\cdot|\eta^k)$.

Proof. We have that

$$\begin{aligned} z(P^*|\eta^k) &= -z(P^k|\eta) + \{z(P^*|\eta^k) + z(P^k|\eta)\} \\ &= -\partial D(P^*) \cdot (P^k - P^*) \\ &\quad +o(\|P^k - P^*\|) \\ &\quad +\{z(P^*|\eta^k) + z(P^k|\eta)\}. \end{aligned}$$

Multiplying by $\sqrt{k}$ and taking the limit in distribution, the $o(\|P^k - P^*\|)$ term in the RHS vanishes, because it is insignificant compared to the first term. Likewise, the third term in the RHS vanishes, by Claim (G1) stated below. Therefore, the equation becomes

$$\sqrt{k}(P^k - P^*) + \partial D(P^*)^{-1} \cdot (\sqrt{k} \cdot z(P^*|\eta^k)) \xrightarrow{d} 0.$$

The result then follows from the observation that $D(P^*|\eta^k)$ has a multinomial distribution with covariance matrix Σ^D. By the central limit theorem we have that $\sqrt{k}\cdot z(P^*|\eta^k)$ is asymptotically normal with covariance matrix Σ^D and 0 mean.

All that remains is to prove the following claim.

Claim G1. We have that

$$\sqrt{k} \cdot \{z(P^*|\eta^k) + z(P^k|\eta)\} \xrightarrow{d} 0. \tag{13}$$

Proof. Define the function

$$\Delta^k(\bar{P}) = \|\{D(\bar{P}|\eta^k) - D(P^*|\eta^k)\} - \{D(\bar{P}|\eta) - D(P^*|\eta)\}\|,$$

which for every $\bar{P}$ and k outputs a random variable. Note that random variable equals the norm of the difference between the realized and expected number of types in certain sets, like empirical and actual distribution functions, so that we can use VC theory to bound it uniformly. Define the ball

$$B_\delta = \{\bar{P} : \|\bar{P} - P^*\| \leq \delta\}.$$

Let the set of students who can potentially change their demand for cutoffs in this ball equal

$$M_\delta = \{\theta \in \Theta : \exists \bar{P} \in B^\delta \text{ and } c \text{ such that } \bar{P}_c = e_c^\theta\}.$$

Finally, define the random variable N_δ^k equal to the number of these potentially marginal students, that is

$$N_\delta^k = \eta^k(M_\delta) \cdot k.$$

Part 1. We will show that, for any $\epsilon, \delta > 0$, the expression

$$F^k(\delta, \epsilon) = \Pr\{\sup_{\bar{P} \in B_\delta} \Delta^k(\bar{P}) \geq k^{-1/2} \cdot \epsilon\}$$

is bounded by

$$F^k(\delta, \epsilon) \leq \alpha \cdot \exp\{-\frac{1}{8} \cdot \epsilon^2 \cdot \frac{1}{\eta(M_\delta) + k^{-1/4}}\} + \exp\{-2 \cdot k^{1/2}\}. \tag{14}$$

To see this, we will first bound the conditional probability

$$\Pr\{\sup_{\bar{P} \in B_\delta} \Delta^k(\bar{P}) \geq k^{-1/2} \cdot \epsilon | N_\delta^k = n\}.$$

Note that $\Delta(\bar{P})$ equals N_δ / k times the norm of the difference between the realized empirical and ex ante distributions of students in a subset of M_δ, conditional on N_δ agents being drawn within M_δ. Hence, by the Vapnik-Chervonenkis Theorem,[13] there exists a constant α such that

$$\begin{aligned} \Pr\{\sup_{\bar{P} \in B_\delta} \Delta(\bar{P}) \geq k^{-1/2} \cdot \epsilon | N_\delta^k = n\} &\leq \alpha \cdot \exp\{-\frac{n}{8} \cdot (k^{-1/2} \cdot \epsilon)^2 \cdot (\frac{k}{n})^2\} \\ &= \alpha \cdot \exp\{-\frac{1}{8} \cdot \epsilon^2 \cdot \frac{k}{n}\}. \end{aligned} \tag{15}$$

To get an unconditional bound, note that, by Hoeffding's inequality,

$$\Pr\{\frac{N_\delta^k}{k} \geq \eta(M_\delta) + k^{-1/4}\} \leq \exp\{-2 \cdot k^{-1/2} \cdot k\}. \tag{16}$$

Combining the bounds (15) and (16), we get the desired inequality (14).

Part 2. We will now use the bound from part 1 to complete the proof.

Note that, because $D(P^*|\eta) = D(P^k|\eta^k) = 0$, we have that

$$\Delta^k(P^k) = \|\{D(P^*|\eta^k) + D(P^k|\eta)\}\|.$$

Using the definition of $F^k(\delta, \epsilon)$, we have that

$$\begin{aligned} \Pr\{\sqrt{k} \cdot \|\{D(P^*|\eta^k) + D(P^k|\eta)\}\| \geq \epsilon\} &= \Pr\{\sqrt{k} \cdot \Delta^k(P^k) \geq \epsilon\} \\ &\leq F^k(\|P^k - P^*\|, \epsilon). \end{aligned}$$

Note that, because P^k converges almost surely to P^* and the strict preferences assumption, we have that $\eta(M_{\|P^k - P^*\|})$ converges almost surely to 0. Using the bound 14, and the fact that $\epsilon > 0$ was taken arbitrarily, we have that the expression in the left-hand side converges to 0 in probability. In particular, it converges to 0 in distribution. That

[13] See Theorem 12.5 in Devroye et al. (1996) p. 197 and the notes in the proof of Proposition 3.

is,

$$\sqrt{k} \cdot \{D(P^*|\eta^k) + D(P^k|\eta)\} \xrightarrow{d} 0.$$

To establish the result, note that, because $\|S^k - S\| \leq 1/k$, we have

$$\|\{z(P^*|\eta^k) + z(P^k|\eta)\} - \{D(P^*|\eta^k) + D(P^k|\eta)\}\| \leq 1/k.$$

This implies the desired result, equation (13). □

□

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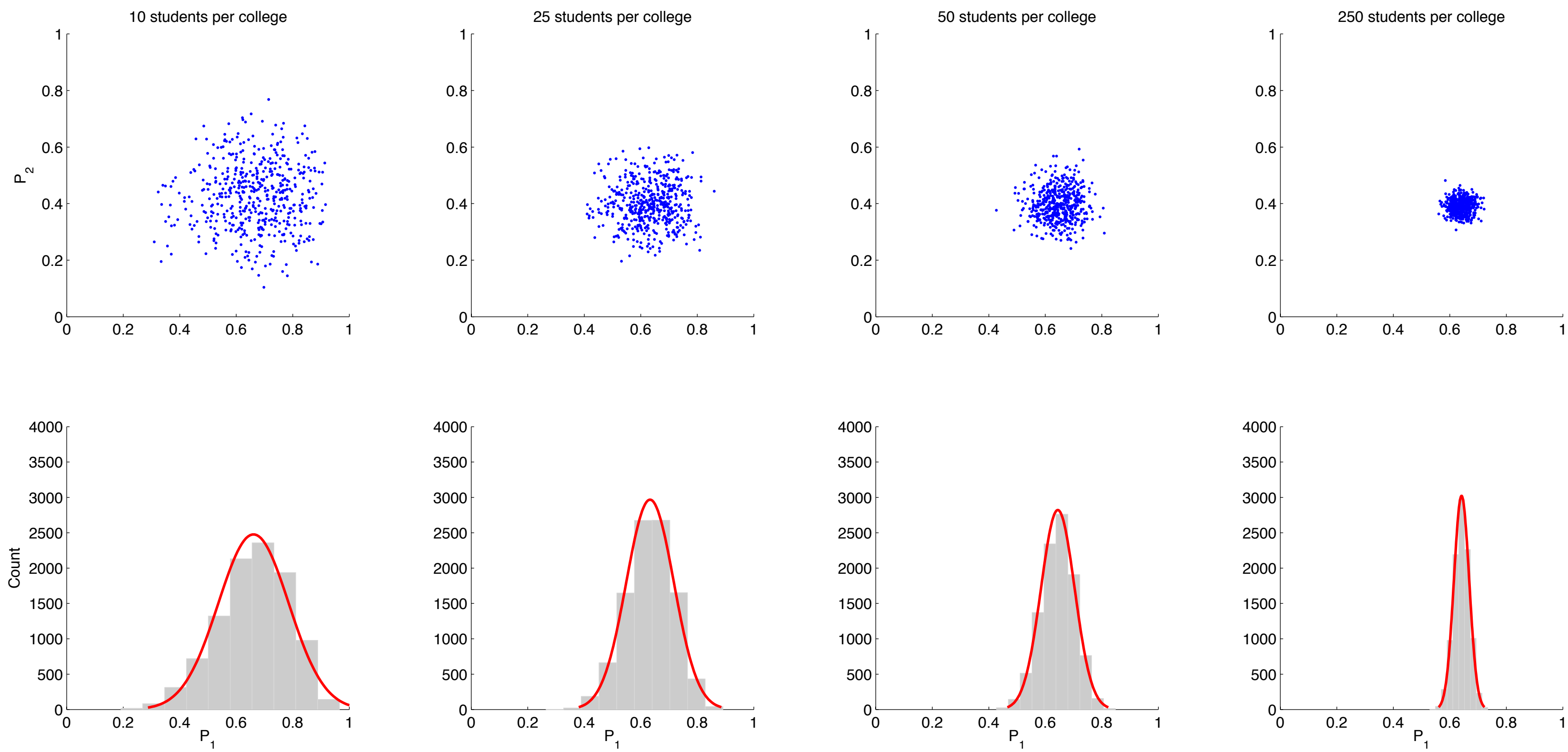

FIGURE G1. Distribution of market clearing cutoffs P^k

Notes: This picture reports realized market clearing cutoffs corresponding to the student-optimal stable matching in 10,000 discrete economies drawn with the distribution of the example in Section 2.3. The top panels display a scatterplot of the joint distribution of market clearing cutoffs in 500 economies. The bottom panels display the histogram of the distribution of the cutoff of college 1, along with a best-fitting normal distribution. The vertical axis in the bottom panels is in units of the count of the histogram bins, and the normal distributions were scaled accordingly.

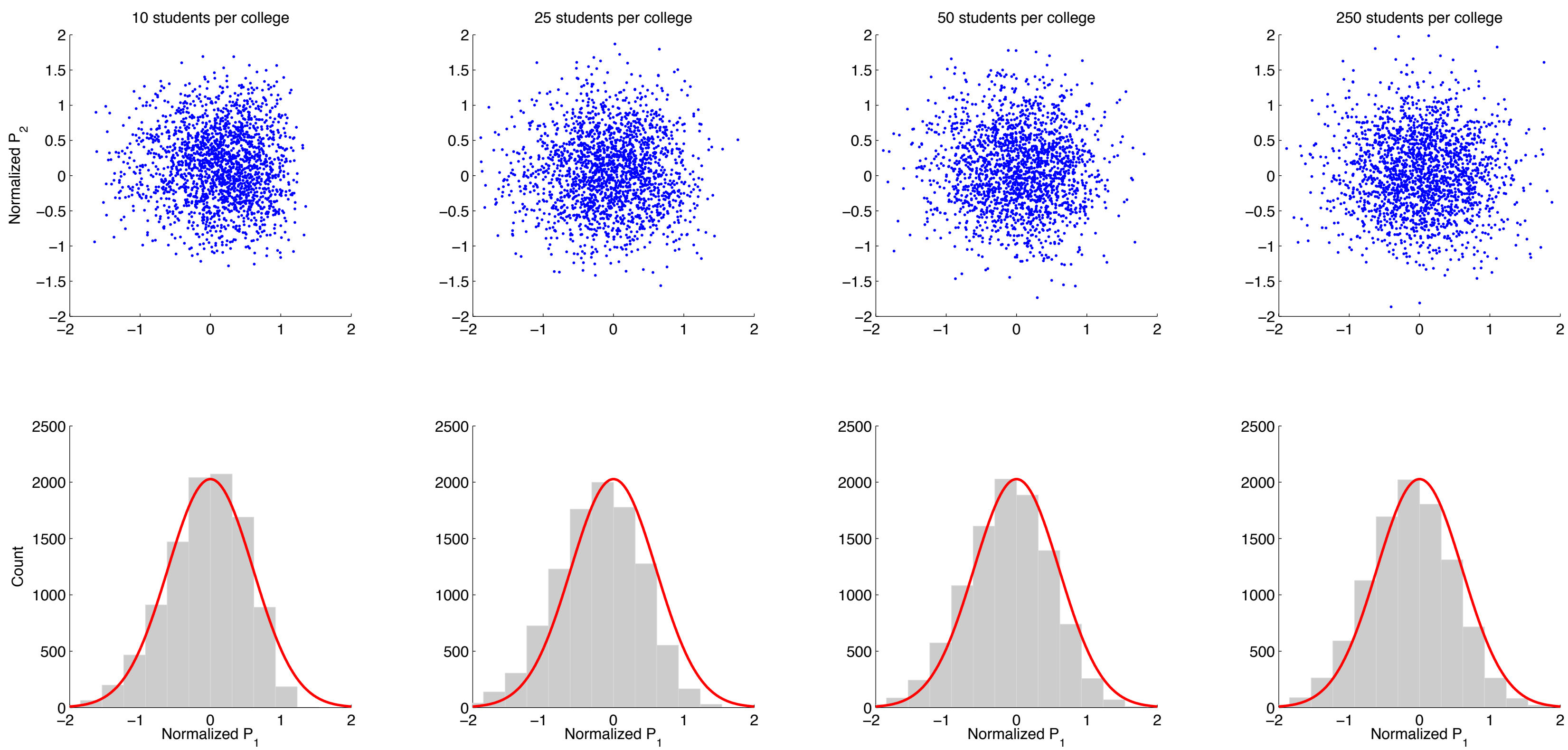

FIGURE G2. Normalized distribution of market clearing cutoffs $\sqrt{k}(P^k - P^*)$

Notes: This picture reports realized market clearing cutoffs corresponding to the student-optimal stable matching in 10,000 discrete economies drawn with the distribution of the example in Section 2.3, normalized by $\sqrt{k}(P^k - P^*)$. The top panels display a scatterplot of the joint distribution of market clearing cutoffs in 2,000 economies. The bottom panels display the histogram of the distribution of the cutoff of college 1. The histogram, which is calculated from the simulations, is overlaid with the normal distribution predicted by Proposition G1. The vertical axis in the bottom panels is in units of the count of the histogram bins, and the normal distribution was scaled accordingly.