Broad Validity of the First-Order Approach in Moral Hazard

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The Moral Hazard Problem (1970s)

- Principal hires agent. Output

$$y \sim f(y|a)$$

- Agent utility

$$u(\text{payment}) - c(a)$$

- Principal's problem is to find optimal contract $w(y)$.

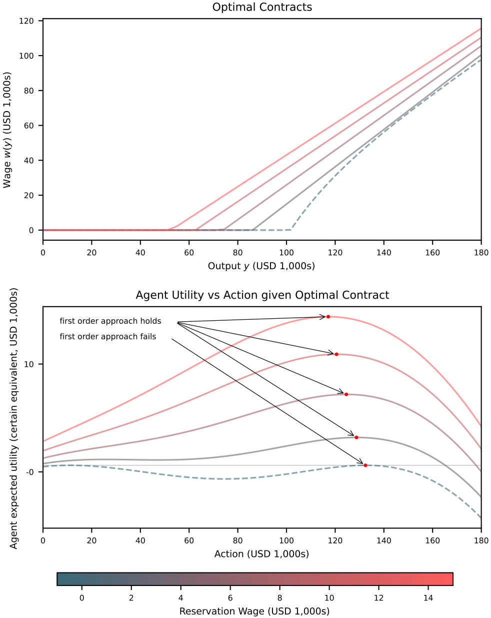
What is known

- The moral hazard problem has a simple solution if the first-order approach is valid (Holmstrom 78).
- Unfortunately, FOA is only valid under restrictive assumptions (Mirrlees 75, Rogerson 85, Jewitt 88, Sinclair-Desgagné 94).
- So theoretical optimal contracts are intractable, complex, and not realistic.
- Motivates huge literature proving linearity. Theoretical literature often assumes FOA anyway. Applied literature often assumes two possible outputs or linearity.

Simplest example

- Gaussian output: $y \sim \mathcal{N}(a, \sigma^2)$, $\sigma = \$50{,}000$.
- Log utility, initial wealth $\$50{,}000$.
- Limited liability.
- Quadratic cost.

Results at odds with received wisdom.



What is actually true

- The FOA holds except for very low reservation utility.
- Key theorem assumptions: reservation utility is high enough and score is increasing.
- Side effects:
    - Tractable (analytically and computationally trivial).
    - Simple (intuitive formula, linear w/ log utility and exponential family with linear sufficient statistic).
    - Realistic.
- Interactive examples: `https://eduardomazevedo.github.io/azevedo-wolff-2025-site/`.

Literature is mathematically correct

- There is not a mathematical error in key prior results.
- The literature is correct that any sufficient condition that guarantees that FOA holds for any reservation utility is extremely restrictive. Gaussian example above would be ruled out because FOA fails at monopsony reservation utility.
- But sufficient conditions are mild if we include sufficiently high reservation utility. See all the examples above and theorem below.

Roadmap

- Model
- Main result
- Applications

Model

Agent utility $u(x) - c(a)$ from payment $x \geq 0$ and action $a \in \mathcal{A}$.
Reservation utility $\bar{U}$.

Output y has density $f(y|a)$.

Contract is $v\colon \mathbb{R} \to u(\mathbb{R}_+)$.

Let $k = u^{-1}$. Feasible contracts: $\mathcal{C}$.

$$U(v, a) = \int \quad v(y) \quad \cdot f(y|a)\,\mathrm{d}y - c(a) \tag{1}$$

$$W(v, a) = \int k(v(y)) \cdot f(y|a)\,\mathrm{d}y \tag{2}$$

Definitions: FOA

GIC (global incentive compatibility): choosing a_0 is optimal for the agent.

LIC (local incentive compatibility): $\partial_a U(v, a_0) = 0$.

Cost minimization problem: minimize expected wage subject to IR and GIC.

Relaxed cost minimization problem: minimize expected wage subject to IR and LIC.

FOA holds for $(a_0, \bar{U})$ if the solution to the relaxed cmp solves the cmp.

Key Assumptions

- Increasing score $S(y|a) := \frac{\partial}{\partial a} \log f(y|a)$.
- Limited liability + limits to punishments: $u(0)$ and $u'(0)$ are finite.
- Need some risk aversion: k'^{-1} is concave.
- Support of $f(y|a)$ is the real line and image of the score is the real line.

Outline

- Model
- Main result
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Theorem 1

Given an action $a_0 > 0$, the first-order approach is valid for $(a_0, \bar{U})$ for any sufficiently high reservation utility $\bar{U}$.

Proof part 1: Relaxed problem

Write Lagrangian:

$$\mathcal{L}(v, \lambda, \mu) = W(v) + \lambda(\bar{U} - U(v)) + \mu(-U_a(v))$$

$$W(v) = \int k(v(y)) \cdot f \; (y|a_0) \, \mathrm{d}y \tag{3}$$

$$U(v) = \int \quad v(y) \quad \cdot f \; (y|a_0) \, \mathrm{d}y - c(a_0) \tag{4}$$

$$U_a(v) = \int \quad v(y) \quad \cdot f_a(y|a_0) \, \mathrm{d}y - c'(a_0) \tag{5}$$

Proof part 1: Relaxed problem

FOC:

$$k'(v(y)) = \lambda + \mu S(y|a_0)$$

Solve for $v(y)$:

$$v(y) = g\left(\lambda + \mu S(y|a_0)\right),$$

where

$$g(z) := k'^{-1}\left(\max\left\{\frac{1}{u'(0)}, z\right\}\right).$$

Proposition 1 ($\approx$ Holmstrom 78)

The relaxed problem has an almost everywhere unique solution $v^*(y|\bar{U})$. There exist $\lambda^*(\bar{U})$, $\mu^*(\bar{U})$ such that

$$v^*(y|\bar{U}) = g\left(\lambda^*(\bar{U}) + \mu^*(\bar{U})S(y|a_0)\right)$$

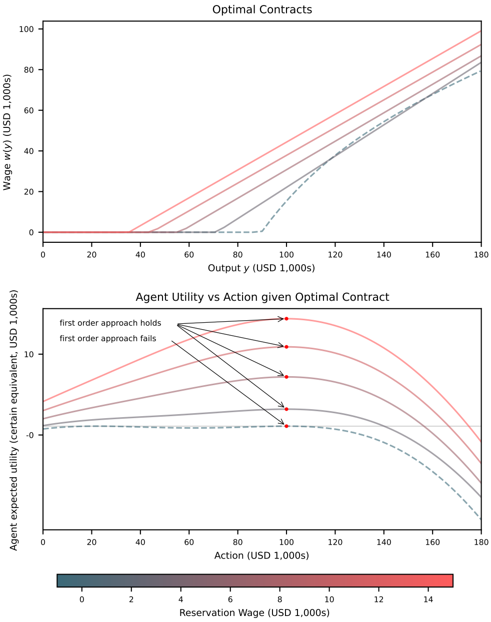
Proof part 2: concavity

Proposition 2:

There exists $U^* \in \mathbb{R}$ such that, for all $\bar{U} \geq U^*$ and all $a \in \mathcal{A}$,

$$U_{aa}(v^*(y|\bar{U}), a) \leq 0.$$

Key idea: kink moves to the left when $\bar{U}$ goes up.



Heuristic derivation

$$U(v, a) = g(\lambda) + \mu \int \Delta g(y|\lambda) \cdot S(y|a_0) \cdot f(y|a) \, \mathrm{d}y - c(a),$$

where

$$\Delta g(y|\lambda) = \frac{g(\lambda + \mu S(y|a_0)) - g(\lambda)}{\mu S(y|a_0)}.$$

So

$$c'(a_0) = \mu \int \Delta g(y|\lambda) \cdot S(y|a_0) \cdot f_a(y|a_0) \, \mathrm{d}y.$$

S and f_a have the same sign. So

$$c'(a_0) \geq \mu \int_{\text{left of the kink}} \Delta g \cdot S \cdot f_a \, \mathrm{d}y.$$

Heuristic derivation

Substituting Δg we get

$$c'(a_0) \geq \int_{\text{left of the kink}} (g(\lambda + \mu S(y|a_0)) - g(\lambda)) \cdot f_a(y|a_0)\,\mathrm{d}y$$
$$= (u(0) - g(\lambda)) \int_{\text{left of the kink}} f_a(y|a_0)\,\mathrm{d}y$$
$$= (u(0) - g(\lambda)) \cdot \frac{\partial F(\underline{y}|a_0)}{\partial a}.$$

Thus, as $\lambda \to \infty$, the left of the kink approaches $-\infty$. Concavity then follows with argument similar to (Jewitt 88).

Outline

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Calculus

- Optimal contracts have simple and intuitive closed-form solutions.
- Calculus of moral hazard has two parts: link function and score.

Table 1: Utility Functions, Link Functions, and Wage Functions

	Utility Function $u(x)$	Link Function $g(z)$	Wage Function $k \circ g(z)$
Log	$\log(x + w_0)$	$\log(\max(w_0, z))$	$(z - w_0)^+$
CRRA	$\frac{(x+w_0)^{1-\gamma}}{1-\gamma}$	$\frac{\max(w_0^\gamma, z)^{\frac{1-\gamma}{\gamma}}}{1-\gamma}$	$\left((z^+)^{\frac{1}{\gamma}} - w_0\right)^+$
CARA	$\frac{-\exp(-\alpha(x+w_0))}{\alpha}$	$-\frac{1}{\alpha \max(\exp(\alpha w_0), z)}$	$\frac{(\log^+ z - \alpha w_0)^+}{\alpha}$

Note: The utility function is the agent's utility from consumption given starting wealth w_0 and a transfer x.

Table 2: Error Distributions

Distribution	Probability Density	Score Function $S(y\|a)$	Mean
Gaussian	$\mathcal{N}(a, \sigma^2)$	$\frac{y-a}{\sigma^2}$	a
Log Normal	$\frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(y)-a)^2}{2\sigma^2}\right)$	$\frac{\log(y)-a}{\sigma^2}$	$\exp\left(a + \frac{\sigma^2}{2}\right)$
Poisson	$\frac{a^y e^{-a}}{y!}$	$\frac{y-a}{a}$	a
Exponential	$\frac{1}{a} e^{-\frac{y}{a}}$	$\frac{y-a}{a^2}$	a
Bernoulli	$a^y(1-a)^{1-y}$, $y \in \{0, 1\}$	$\frac{y-a}{a-a^2}$	a
Geometric	$\left(1 - \frac{1}{a}\right)^{y-1}\left(\frac{1}{a}\right)$, $y \in \{1, 2, \dots\}$	$\frac{y-a}{a^2-a}$	a
Binomial	$\binom{n}{y} a^y (1-a)^{n-y}$, $y \in \{0, \dots, n\}$	$\frac{y-na}{a-a^2}$	na
Gamma	$f(y \mid n, a) = \frac{y^{n-1} e^{-\frac{y}{a}}}{\Gamma(n)\, a^n}$	$\frac{y-na}{a^2}$	na
Student's t	$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\,\sigma} \left(1 + \frac{1}{\nu} \frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+1}{2}}$	$\frac{(\nu+1)(y-a)}{\nu\,\sigma^2+(y-a)^2}$	a
Exponential Family	$h(y) \exp\left(\eta(a)\, T(y) - A(a)\right)$	$T(y)\, \frac{d\eta(a)}{da} - \frac{dA(a)}{da}$	*(Not specified)*
$y = a + X, X \sim h$	$g(y-a)$	$-\frac{g'(y-a)}{h(y-a)}$	$a + \mathbb{E}[X]$
$y = aX, X \sim h$	$\left\|\frac{1}{a}\right\| h\left(\frac{y}{a}\right)$	$-\frac{1}{a} - \frac{y}{a^2} \frac{g'(\frac{y}{a})}{g(\frac{y}{a})}$	$a\, \mathbb{E}[X]$

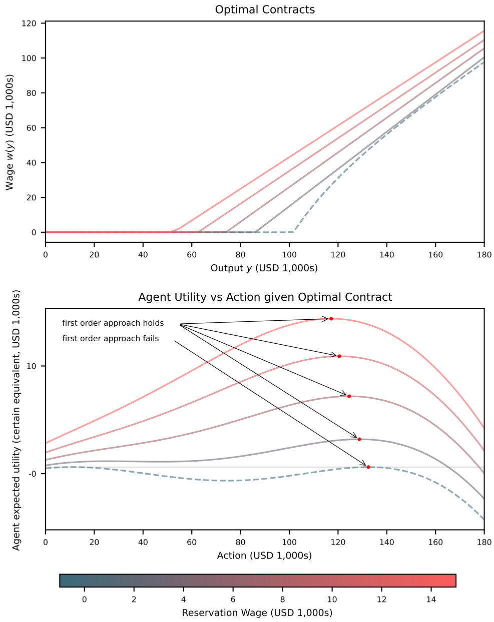
Note: This table presents probability the PDF, score function, and means of probability distributions as functions of the agent's chosen action, a.

Piecewise linear contracts

Optimal contract is piecewise linear for log utility + exponential family with linear sufficient statistic.

Prior results: Opp 25 (independently), earlier papers without existence (Hemmer 99, Christensen 05).

Global linearity in some cases.



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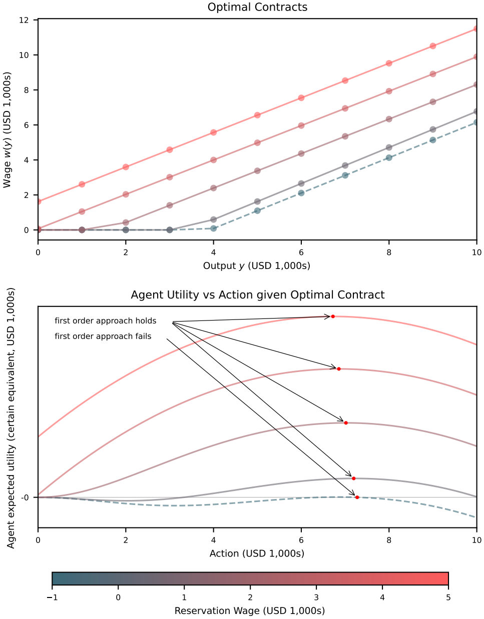


Numerical methods

- Results suggest efficient algorithms.
- Consider a finite set of GIC constraints at $\hat{a} = (\hat{a}_1, \dots, \hat{a}_n)$.
- With multipliers $\theta := (\lambda, \mu, \hat{\mu})$, the optimal contract is

$$V(y|\theta, \hat{a}) = g\left(\lambda + \mu S(y|a_0) + \sum_i \hat{\mu}_i \left(1 - \frac{f(y|\hat{a}_i)}{f(y|a_0)}\right)\right).$$

- Lagrange dual $\mathcal{D}(\theta, \hat{a})$ is fast to compute if we cache these terms.
- Analytic gradient from Danskin's envelope theorem.

Numerical methods: algorithm

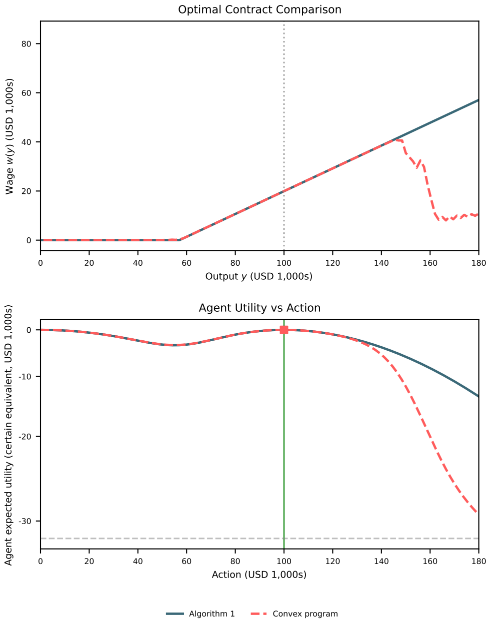
1. Start with $\hat{a} = \emptyset$.
2. Maximize dual to solve relaxed problem, and add best deviation action to $\hat{a}$ if needed.
3. Repeat until no deviation is found.
4. In case of trouble, fall back to convex optimizer to find more deviations for $\hat{a}$.

Performance

Active-set algorithm runs in about 3–14 ms (CMP) and 232–240 ms (PP) vs. hours reported in the literature.

Why it is fast:

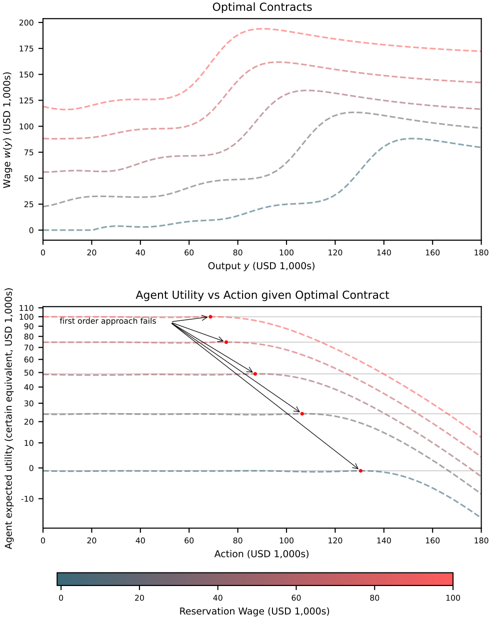
- Often terminates in one iteration.
- Dual calls use analytic formulas + caching.
- Low-dimensional dual with analytic gradients + warm starts.



Key assumption: monotone score.

Example: t-distributed outcome, tail coefficient $\nu = 1.15$.

Score is not monotone, and FOA often fails even for high reservation utility.



Conclusion

- FOA is usually valid, we just have to assume high reservation utility.
- Tractable, intuitive, realistic solutions—sometimes linear contracts.

Thank you!