Appendix: Functional Form Derivations

1 Distributions

1.1 Normal

The PDF is

$$f(y|a,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right).$$

The derivative with respect to a is

$$f_a(y|a,\sigma^2) = f(y|a,\sigma^2)\frac{y-a}{\sigma^2}.$$

The score is

$$S(y|a) = \frac{f(y|a,\sigma^2)\frac{y-a}{\sigma^2}}{f(y|a,\sigma^2)} = \frac{y-a}{\sigma^2}.$$

1.2 Log Normal

The PDF is

$$f(y|a,\sigma^2) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(y)-a)^2}{2\sigma^2}\right).$$

The derivative with respect to a is

$$f_a(y|a,\sigma^2) = f(y|a,\sigma^2)\frac{log(y)-a}{\sigma^2}.$$

The score is

$$S(y|a) = \frac{f(y|a,\sigma^2)\frac{log(y)-a}{\sigma^2}}{f(y|a,\sigma^2)} = \frac{log(y)-a}{\sigma^2}.$$

1.3 Poisson

The PMF is

$$P(y|a) = \frac{a^y e^{-a}}{y!}.$$

The derivative with respect to a is

$$P_a(y|a) = \frac{y a^{y-1} e^{-a}}{y!} - \frac{a^y e^{-a}}{y!}.$$

Simplifying yields

$$P_a(y|a) = P(y|a)(\frac{y}{a} - 1) = f(a)\frac{y-a}{a}.$$

The score is

$$S(y|a) = \frac{f(a)\frac{y-a}{a}}{f(a)} = \frac{y-a}{a}.$$

1.4 Exponential

The PDF is

$$f(y|a) = \frac{1}{a}e^{-\frac{y}{a}}.$$

Observe that $\frac{\partial a}{\partial}\frac{1}{a} = -\frac{1}{a^2}$, and $\frac{\partial a}{\partial}e^{-\frac{y}{a}} = \frac{y}{a^2}e^{-\frac{y}{a}}$. Using the product rule,

$$f_a(y|a) = \frac{y}{a^3}e^{-\frac{y}{a}} - \frac{1}{a^2}e^{-\frac{y}{a}}.$$

Simplifying yields

$$f_a(y|a) = \frac{y-a}{a^2}f(y|a).$$

The score is

$$S(y|a) = \frac{y-a}{a^2}.$$

1.5 Bernoulli

The PMF is

$$P(y|a) = a^y(1-a)^{1-y},\ where\, y \in \{0,1\}.$$

The derivative with respect to a is

$$P_a(y|a) = ya^{y-1}(1-a)^{1-y} - (1-y)a^y(1-a)^{-y}.$$

Simplifying yields

$$P_a(y|a) = P(y|a)(\frac{y}{a} - \frac{1-y}{1-a}) = P(y|a)\frac{y-a}{a-a^2}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{y-a}{a-a^2}}{P(y|a)} = \frac{y-a}{a-a^2}.$$

1.6 Geometric

The PMF is

$$P(y|a) = (1 - \frac{1}{a})^{y-1}(\frac{1}{a}).$$

The derivative with respect to a is

$$P_a(y|a) = (y-1)(\frac{a-1}{a})^{y-2}(\frac{1}{a^3}) - (1 - \frac{1}{a})^{y-1}(\frac{1}{a^2}).$$

Simplifying yields

$$P_a(y|a) = \frac{(y-1)a}{a-1}(\frac{a-1}{a})^{y-1}\frac{1}{a^3} - P(y|a)\frac{1}{a}.$$

We simplify further

$$P_a(y|a) = \frac{(y-1)}{(a-1)a}P(y|a) - P(y|a)\frac{1}{a}.$$

We factor out $P(y|a)$

$$P_a(y|a) = P(y|a)\frac{(y-a)}{(a^2-a)}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{(y-a)}{(a^2-a)}}{P(y|a)} = \frac{y-a}{a^2-a}$$

1.7 Binomial

The PMF is

$$P(y|a,n) = \binom{n}{y} a^y (1-a)^{n-y}.$$

The derivative with respect to a is

$$P_a(y|a) = y\binom{n}{y} a^{y-1}(1-a)^{n-y} - (n-y)\binom{n}{y} a^y (1-a)^{n-y-1}.$$

Simplifying yields

$$P_a(y|a) = \frac{y}{a}P(y|a,n) - \frac{n-y}{1-a}P(y|a,n).$$

I factor out $P(y|a)$ and simplify further

$$P_a(y|a) = P(y|a,n)\frac{y-na}{a-a^2}.$$

The score is

$$S(y|a) = \frac{P(y|a)\frac{(y-na)}{(a-a^2)}}{P(y|a)} = \frac{y-na}{a-a^2}.$$

1.8 Gamma

The PDF is

$$f(y|n,a) = \frac{y^{n-1}e^{-y/a}}{\Gamma(n)a^n}.$$

The derivative with respect to a is

$$f_a(y|n,a) = \frac{y}{a^2}f(y|n,a) - \frac{n}{a}f(y|n,a).$$

I factor out $f(y|n,a)$ and simplify

$$f_a(y|n,a) = f(y|n,a)\frac{y-na}{a^2}.$$

The score is

$$S(y|a) = \frac{f(y|n,a)\frac{y-na}{a^2}}{f(y|n,a)} = \frac{y-na}{a^2}.$$

1.9 t

The PDF is

$$f(y|\nu,a,\sigma) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+1}{2}}$$

The derivative with respect to a is

$$f_a(y|\nu,a,\sigma) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\sigma}\frac{\nu+1}{2}\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)^{-\frac{\nu+3}{2}}\frac{2(y-a)}{\nu\sigma^2}.$$

Simplifying yields

$$f_a(y|\nu,a,\sigma) = f(y|\nu,a,\sigma)\frac{(\nu+1)(y-a)}{\nu\sigma^2\left(1+\frac{1}{\nu}\frac{(y-a)^2}{\sigma^2}\right)} = f(y|\nu,a,\sigma)\frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}.$$

The score is

$$S(y|a) = \frac{f(y|n,a)\frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}}{f(y|n,a)} = \frac{(\nu+1)(y-a)}{\nu\sigma^2+(y-a)^2}.$$

1.10 Exponential Family

The PDF is

$$f(y|a) = h(y)\exp\left(\eta(a)T(y) - A(a)\right),$$

where $h(y)$ is the base measure, $\eta(a)$ is the natural parameter, $T(y)$ is the sufficient statistic, and $A(a)$ is the log-partition function. The derivative with respect to a is

$$f_a(y|a) = h(y)\exp\left(\eta(a)T(y) - A(a)\right)\left[T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}\right].$$

Simplifying yields

$$f_a(y|a) = f(y|a)\left[T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}\right].$$

The score is

$$S(y|a) = \frac{f_a(y|a)}{f(y|a)} = T(y)\frac{d\eta(a)}{da} - \frac{dA(a)}{da}.$$

1.11 Additive

Let $y = a + y$, where y is a random variable with PDF $g(y)$. The PDF of y is

$$f(y|a) = g(y - a).$$

The derivative with respect to a is

$$f_a(y|a) = -g'(y - a).$$

The score is

$$S(y|a) = \frac{g'(y-a)}{g(y-a)}.$$

1.12 Multiplicative

Let $y = ay$ where y is a random variable with PDF $g(y)$. The PDF of y is

$$f(y|a) = \frac{1}{a}g(\frac{y}{a}).$$

The derivative with respect to a is

$$-\frac{1}{a^2}g(\frac{y}{a}) - \frac{1}{a^3}g'(\frac{y}{a})y.$$

The score is

$$S(y|a) = -\frac{1}{a} - \frac{1}{a^2}\frac{g'(\frac{y}{a})}{g(\frac{y}{a})}y$$

2 Utility Functions

2.1 Log

This section derives results for the utility function

$$u(x) = log(w_0 + x).$$

The derivative of utility with respect to consumption is

$$u'(x) = \frac{1}{w_0 + x}$$

The cost of utility is the inverse of the utility function

$$u = log(w_0 + x) \rightarrow exp(u) = x \rightarrow k(u) = exp(u) - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{\frac{1}{exp(u)}} = exp(u).$$

The inverse of the marginal cost of utility is

$$x = exp(u) \rightarrow k'^{-1}(x) = log(x).$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = \log\left(\max(w_0, z)\right).$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned} w(z) &= k(g(z)) \\ &= \exp\big(\log(max(w_0, z))\big) - w_0 \\ &= (z - w_0)^+ . \end{aligned}$$

2.2 CRRA

This section derives results for the utility function

$$u(x) = \frac{(w_0 + x)^{1-\gamma}}{1 - \gamma}.$$

The derivative of utility with respect to consumption is

$$u'(x) = (w_0 + x)^{-\gamma}.$$

The cost of utility is the inverse of the utility function

$$u = \frac{(w_0 + x)^{1-\gamma}}{1-\gamma} \to k(u) = ((1-\gamma)u)^{\frac{1}{1-\gamma}} - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{((1-\gamma)u)^{\frac{-\gamma}{1-\gamma}}} = ((1-\gamma)u)^{\frac{\gamma}{1-\gamma}}.$$

The inverse of the marginal cost of utility is

$$x = ((1-\gamma)u)^{\frac{\gamma}{1-\gamma}} \to k'^{-1}(x) = \frac{x^{\frac{1-\gamma}{\gamma}}}{1-\gamma}.$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = \frac{\max\left(w_0^{\gamma}, z\right)^{\frac{1-\gamma}{\gamma}}}{1-\gamma}.$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned} w(z) &= k(g(z)) \\ &= \left(\frac{1-\gamma}{1-\gamma} \cdot \max(w_0^{\gamma}, z)^{\frac{1-\gamma}{1-\gamma}}\right)^{\frac{1}{1-\gamma}} - w_0 \\ &= max(w_0^{\gamma}, z)^{\frac{1}{\gamma}} - w_0 \\ &= \left((z^+)^{\frac{1}{\gamma}} - w_0\right)^+. \end{aligned}$$

2.3 CARA

This section derives results for the utility function

$$u(x) = \frac{-\exp(-\alpha(x + w_0))}{\alpha}.$$

The derivative of utility with respect to consumption is

$$u'(x) = exp(-\alpha(x + w_0)).$$

The cost of utility is the inverse of the utility function

$$u = \frac{-\exp(-\alpha(x + w_0))}{\alpha} \to -\alpha u = exp(-\alpha(x + w_0)) \to k(u) = -\frac{log(-\alpha u)}{\alpha} - w_0.$$

The marginal cost of utility is

$$\frac{1}{u'(k(u))} = \frac{1}{exp(\alpha\frac{log(-\alpha u)}{\alpha})} \to k'(u) = -\frac{1}{\alpha u}.$$

The inverse of the marginal cost of utility is

$$x = -\frac{1}{\alpha u} \rightarrow k'^{-1}(x) = -\frac{1}{\alpha x}.$$

The link function, $g(z)$, is

$$g(z) = k'^{-1}\left(\max\left(\frac{1}{u'(w_0)}, z\right)\right) = -\frac{1}{\alpha \max\left(\exp(\alpha w_0), z\right)}.$$

The canonical wage function given $z = \lambda + \mu S(y|a_0)$ is

$$\begin{aligned}
w(z) &= k(g(z)) \\
&= k\left(-\frac{1}{\alpha \max\left(\exp(\alpha w_0), z\right)}\right) \\
&= -\frac{\log\left(-\alpha \cdot \left(-\frac{1}{\alpha \max(\exp(\alpha w_0), z)}\right)\right)}{\alpha} - w_0 \\
&= -\frac{(log(1) - \log\left(\max\left(\exp(\alpha w_0), z\right)\right))}{\alpha} - w_0 \\
&= \frac{\log\left(\max\left(\exp(\alpha w_0), z\right)\right)}{\alpha} - w_0 \\
&= \frac{(\log^+ z - \alpha w_0)^+}{\alpha}.
\end{aligned}$$